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Exact solutions for weakly singular Volterra integral equations by using an efficient iterative method

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Abstract : *In this paper, an iterative method proposed by Daftardar-Gejji and Jafari namely (DJM) will be presented to solve the weakly singular Volterra integral equation (WSVIE) of the second kind. This method is able to solve large class of linear and nonlinear equations effectively, more accurately and easily. In this iterative method the solution is obtained in the series form that converges to the exact solution if it exists. The main contribution of the current paper is to obtain the exact solution rather than numerical solution as done by some existing techniques. The results demonstrate that the method has many merits such as being derivative-free, overcome the difficulty arising in calculating Adomian polynomials to handle the nonlinear terms in Adomian Decomposition Method (ADM). It does not require to calculate Lagrange multiplier as in Variational Iteration Method (VIM) and no needs to construct a homotopy and solve the corresponding algebraic equations as in Homotopy Perturbation Method (HPM). The results reveal that the method is accurate and easy to implement. The software used for the calculations in this study was MATHEMATICA® 10.0.*

Keywords - *Volterra integral equation; Singular kernel; Exact solution; Iterative method.*

I. Introduction

A variety of problems in the other sciences, such as physics and chemistry have their mathematical setting as integral equations. These problems formulated as an integral equations of two types linear and nonlinear. There are several methods available in iterative to solve these equations which began to take great interest by researchers in recent years. In this work we will study the second type of weakly singular Volterra integral equation (WSVIE) where the kernel becomes infinite at one or more points of singularities at the range of definition. The WSVIE have been investigated by many researchers which has some difficulty to handle [1]. WSVIE appears in applications in mathematical physics and chemical reaction including stereology [2], the radiation of heat from a semi-infinite solid [3] and heat conduction, crystal growth, electrochemistry, superfluidity [4].

In this work, the general form of weakly singular Volterra integral equation of second kind given in [5] will be solved:

$$u(x) = f(x) + \int_0^x k(x,t)u(t)dt \qquad x \in [0, X], \qquad \dots (1)$$

Where $k(x,t) = \frac{t^{\mu-1}}{x^{\mu}}$, $\mu > 0$ and *f(x)* is a given function, $k(x,t)$ is the kernel function, $u(x)$ is the unknown function to be determined.

Now we will study the two cases of μ.

(a) $0 < \mu \leq 1$ if $0 < \mu < 1$ the kernel is singular at $x = 0$ and at $t = 0$, for all values of $t > 0$. Eq. (1) has an infinite set of solutions, but if $\mu = 1$ then the kernel is a singularity only at $x = 0$, and in this case when, $f \in C^1[0, X]$ (with $f(0) = 1$), Eq. (1) has an infinite set of solutions in C [0, X], which contains only one particular solution belonging to $C^1[0, X]$ [6].

(b) $\mu > 1$ the kernel is a singularity only at $x = 0$, in this case the equation (1) has unique solution in $C^m[0, X], f \in C^m[0, X]$ [7].

Many attempted have been proposed to solve Eq. (1) for the case when $\mu > 1$, the well-known methods contain the generalized Newton–Cotes formulae combined with product integration rules [8,9], extrapolation algorithm [10], Hermite type collocation method [11], spline collocation and iterated collocation methods [6,12]. Adomian decomposition method (ADM) [13], the variational iterative method (VIM) [14], the Homotopy perturbation method (HPM) [15–16], the optimal homotopy asymptotic method (OHAM) [17]. Also, the reproducing kernel method is used to solve the WSVIE in [18]. Some of these methods are used to find analytic, approximate and numerical solutions. On the other hand, most of the methods faced difficulty in solving the equation in the case $0 < \mu \leq 1$, for example Nystrom interpolant method [19], extrapolation methods [20], graded mesh method [21].

In this paper, the DJM will be implemented to solve the linear and nonlinear WSVIE. This method developed in 2006 by Daftardar-Gejji and Jafari [22, 23], has been widely efficiency used by researchers to solve linear and nonlinear equations [24, 25, 26, 27].

Recently, AL-Jawary et al. [28–32] have successfully implemented the DJM for solving different linear and nonlinear ordinary and partial differential equations [28–32]. The purpose of this paper is to obtain the analytic solution of linear and nonlinear weakly singular Volterra integral equation of the second kind, rather than numerical solutions as done by some existing techniques.

The present paper has been arranged as follows. In section 2**,** the basic idea of DJM is explained. In section 3, solving weakly singular Volterra integral equation by using DJM is presented. In section 4**,** some test examples are given and finally in section 5 the conclusion is presented.

II. Basic idea DJM

Consider the following general functional equation [22]:

$$u = N(u) + f, \qquad \text{...(2)}$$

Where N is nonlinear operator from a Banach space $B \to B$, u is an unknown function and f is a known function.

We are looking for a solution u of Eq. (2) having the series from:

$$u = \sum_{i=0}^{\infty} u_i \qquad \text{... (3)}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \qquad \text{... (4)}$$

From Eq. (3) and Eq. (4), Eq. (2) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=0}^{\infty} \left\{ N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \qquad \text{... (5)}$$

We define the recurrence relation:

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), \quad m = 1,2,\dots . \end{cases} \qquad \text{... (6)}$$

Then

$$u = f + \sum_{i=0}^{\infty} u_i \, . \qquad \text{... (7)}$$

For the convergence of DJM, we refer the reader to [23].

III. Solving weakly singular Volterra integral equation by using DJM

In this section, the DJM will be implemented to obtain the exact solution for a weakly singular Volterra integral equation of second kind. Let us consider a form of WSVIE given in Eq.(1).

$$u(x) = f(x) + \int_0^x \frac{t^{\mu-1}}{x^{\mu}}\, u(t)dt \qquad x \in [0, X].$$

By using Eqs. (6) and (7), we obtain the following recurrence relation.

$$u_0(x) = f(x), \qquad \text{...(8)}$$

$$u_1(x) = N(u_0) \quad = \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u_0(t)dt \,, \qquad \text{... (9)}$$

$$u_2(x) = N(u_0 + u_1) - N(u_0)$$
$$= \int_0^x \frac{t^{\mu-1}}{x^{\mu}} (u_0 + u_1)(t)dt - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u_0(t)dt \,, \qquad \text{... (10)}$$

$$u_3(x) = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$$
$$= \int_0^x \frac{t^{\mu-1}}{x^{\mu}}(u_0 + u_1 + u_2)(t)d - \int_0^x \frac{t^{\mu-1}}{x^{\mu}}(u_0 + u_1)(t)dt \ , \quad \ldots(11)$$
$$u_i(x) = \int_0^x \frac{t^{\mu-1}}{x^{\mu}}(u_0 + \cdots + u_{i-1})(t)dt - \int_0^x \frac{t^{\mu-1}}{x^{\mu}}(u_0 + \cdots + u_{i-2})(t)dt \ , \quad \ldots(12)$$
Then

$$u(x) = \sum_{i=0}^{\infty} u_i \ . \quad \ldots(13)$$

IV. Test examples

In this section the applications of the DJM for the linear and nonlinear WSVIE will be presented for several examples to assess the efficiency of the proposed method.

4.1 Linear WSVIE

Let us solve first some linear WSVIEs by DJM for two cases:
Case 1: $\mu > 1$:

We applied the DJM to solve the examples for the WSVIE when $\mu > 1$ which already solved numerically by some existing techniques [8, 9, 11], ADM [13], VIM [14], HPM [15–16], OHAM [13], and obtain the exact solutions.

Example 1:

Let us consider the following linear WSVIE [17]:
$$u(x) = \frac{44}{54}\, x^{-0.5} + x^{-5.9}\int_0^x t^{4.9}\, u(t)dt, \quad \ldots(14)$$
Where, $\mu = 5.9$, $f(x) = \frac{44}{54}x^{-0.5}$, and the exact solution is $u(x) = x^{-0.5}$.
By applying the DJM, we obtain:
$$u_0(x) = f(x) = \frac{44}{54}x^{-0.5} = \frac{22}{27\sqrt{x}} \ , \quad \ldots(15)$$
$$u_1(x) = x^{-5.9}\int_0^x t^{4.9}\, u_0(t)dt = \frac{110}{729\sqrt{x}} \ , \quad \ldots(16)$$
$$u_2(x) = x^{-5.9}\int_0^x t^{4.9}\,(u_0 + u_1)(t)dt - u_1(x) = \frac{550}{19683\sqrt{x}} \ , \quad \ldots(17)$$
$$u_3(x) = x^{-5.9}\int_0^x t^{4.9}\,(u_0 + u_1 + u_2)(t)dt - x^{-5.9}\int_0^x t^{4.9}\,(u_0 + u_1)(t)dt = \frac{2750}{531441\sqrt{x}} \ , \quad \ldots(18)$$
and so on.
Continuing in this manner, the approximation of the exact solutions for the unknown functions u(x) can be achieved as:
$$u = u_0 + u_1 + u_2 + u_3 + \cdots$$
$$= \frac{22}{27\sqrt{x}} + \frac{110}{729\sqrt{x}} + \frac{550}{19683\sqrt{x}} + \frac{2750}{531441\sqrt{x}} + \cdots, \quad \ldots(19)$$
We get an infinite geometric series.
$$\sum_{i=0}^{\infty} ar^n = a(1 + r + r^2 + \cdots) \ , \quad \ldots(20)$$
Where $a > 0$ and r is base of series.
if $|r| < 1$, and $n \to \infty$, then
$$\sum_{i=0}^{\infty} ar^n = \frac{a}{1-r} \ , \quad \ldots(21)$$

$$u = x^{-0.5}\left(\frac{\frac{22}{27}}{1-\frac{5}{27}}\right) = x^{-0.5}\ , \qquad \dots(22)$$

Which the exact solution for Eq. (14) [12,17].

Example 2:

Let us consider the following linear WSVIE given in [17]:

$$u(x) = \frac{65}{75}\,x^3 + \frac{7}{8}x^{3.5} + x^{-4.5}\int_0^x t^{3.5}\,u(t)dt, \qquad \dots(23)$$

Where, $\mu = 4.5$, and the exact solution is $u(x) = x^{3.5} + x^3$.
Using the recurrence relation defined in equation Eqs. (6) and (7), we obtain

$$u_0(x) = f(x) = \frac{13}{15}x^3 + \frac{7}{8}x^{3.5}\ , \qquad \dots(24)$$

$$u_1(x) = N(u_0) = x^{-4.5}\int_0^x t^{3.5}\,u_0(t)dt = \frac{26x^3}{225} + \frac{7x^{7/2}}{64}, \qquad \dots(25)$$

$$u_2(x) = x^{-4.5}\int_0^x t^{3.5}(u_0+u_1)(t)dt - u_1(x) = \frac{52x^3}{3375} + \frac{7x^{7/2}}{512}, \qquad \dots(26)$$

$$u_3(x) = x^{-4.5}\int_0^x t^{3.5}(u_0+u_1+u_2)(t)dt - x^{-4.5}\int_0^x t^{3.5}(u_0+u_1)(t)dt = \frac{104x^3}{50625} + \frac{7x^{7/2}}{4096}, \qquad \dots(27)$$

and so on. The solution in a series form is given by:

$$u = u_0 + u_1 + u_2 + u_3 + \cdots$$
$$= \frac{13x^3}{15} + \frac{7x^{7/2}}{8} + \frac{26x^3}{225} + \frac{7x^{7/2}}{64} + \frac{52x^3}{3375} + \frac{7x^{7/2}}{512} + \frac{104x^3}{50625} + \frac{7x^{7/2}}{4096} + \cdots, \qquad \dots(28)$$

$$u = \left(\frac{13x^3}{15} + \frac{26x^3}{225} + \frac{52x^3}{3375} + \frac{104x^3}{50625} + \dots\right) + \left(\frac{7x^{7/2}}{8} + \frac{7x^{7/2}}{64} + \frac{7x^{7/2}}{512} + \frac{7x^{7/2}}{4096} + \dots\right), \qquad \dots(29)$$

$$u = x^3\left(\frac{\frac{13}{15}}{1-\frac{2}{15}}\right) + x^{\frac{7}{2}}\left(\frac{\frac{7}{8}}{1-\frac{1}{8}}\right) = x^3 + x^{3.5}\ , \qquad \dots(30)$$

Which the exact solution for Eq. (23) [12, 17], obtain upon using the sum of infinite geometric series in Eq. (20).

Example 3:

Let us consider the following linear WSVIE given in [17]:

$$u(x) = \frac{55}{65}\,x^5 + \frac{7}{8}x^{6.5} + x^{-1.5}\int_0^x t^{0.5}\,u(t)dt\ , \qquad \dots(31)$$

Where, $\mu = 1.5$, $f(x) = \frac{55}{65}\,x^5 + \frac{7}{8}x^{6.5}$, and the exact solution is $u(x) = x^{6.5} + x^5$.
By applying the DJM, the following recurrence relation for the determination of the components u_{n+1}(x) are obtained:

$$u_0(x) = f(x) = \frac{11}{13}x^5 + \frac{7}{8}x^{6.5}\ , \qquad \dots(32)$$

$$u_1(x) = N(u_0) = x^{-1.5}\int_0^x t^{0.5}\,u_0(t)dt = \frac{22x^5}{169} + \frac{7x^{13/2}}{64}, \qquad \dots(33)$$

$$u_2(x) = x^{-1.5}\int_0^x t^{0.5}(u_0+u_1)(t)dt - u_1(x) = \frac{44x^5}{2197} + \frac{7x^{13/2}}{512}, \qquad \dots(34)$$

$$u_3(x) = x^{-1.5}\int_0^x t^{0.5}(u_0+u_1+u_2)(t)dt - x^{-1.5}\int_0^x t^{0.5}(u_0+u_1)(t)dt$$
$$= \frac{88x^5}{28561} + \frac{7x^{13/2}}{4096}, \qquad \dots(35)$$

and so on. The solution in a series form is given by:

$$u = u_0 + u_1 + u_2 + u_3 + \cdots$$
$$= \frac{11x^5}{13} + \frac{7x^{13/2}}{8} + \frac{22x^5}{169} + \frac{7x^{13/2}}{64} + \frac{44x^5}{2197} + \frac{7x^{13/2}}{512} + \frac{88x^5}{28561} + \frac{7x^{13/2}}{4096} + \cdots, \quad \ldots(36)$$

$$u = \left(\frac{11x^5}{13} + \frac{22x^5}{169} + \frac{44x^5}{2197} + \frac{88x^5}{28561} + \cdots\right) + \left(\frac{7x^{13/2}}{8} + \frac{7x^{13/2}}{64} + \frac{7x^{13/2}}{512} + \frac{7x^{13/2}}{4096} + \cdots\right), \quad \ldots(37)$$

$$u = x^5\left(\frac{\frac{11}{13}}{1-\frac{2}{13}}\right) + x^{\frac{13}{2}}\left(\frac{\frac{7}{8}}{1-\frac{1}{8}}\right) = x^5 + x^{6.5}, \quad \ldots(38)$$

Which the exact solution for Eq. (31) [12, 17], obtain upon using the sum of infinite geometric series in Eq. (20).

Case 2: 0 < μ ≤ 1:

We applied the DJM to solve some examples of the WSVIE when 0 < μ ≤ 1 which already solved numerically by some existing techniques [19, 20, 21], and obtain the exact solution.

Example 4:

Let us consider the following linear WSVIE given in [17]:

$$u(x) = x^{1.5} + x + 1 + x^{-0.5}\int_0^x t^{-0.5}\, u(t)dt, \quad \ldots(39)$$

Where, $\mu = 0.5, f(x) = x^{1.5} + x + 1$, and the exact solution is $u(x) = 2x^{1.5} + 3x - 1$.
By Appling the DJM, we obtain:

$$u_0(x) = f(x) = 1 + x + x^{1.5}, \quad \ldots(40)$$

$$u_1(x) = N(u_0) = x^{-0.5}\int_0^x t^{-0.5}\, u_0(t)dt = 2 + \frac{2x}{3} + \frac{x^{3/2}}{2}, \quad \ldots(41)$$

$$u_2(x) = x^{-0.5}\int_0^x t^{-0.5}(u_0 + u_1)(t)dt - u_1(x) = 4 + \frac{4x}{9} + \frac{x^{3/2}}{4}, \quad \ldots(42)$$

$$u_3(x) = x^{-0.5}\int_0^x t^{-0.5}(u_0 + u_1 + u_2)(t)dt - x^{-0.5}\int_0^x t^{-0.5}(u_0 + u_1)(t)dt$$
$$= 12 + \frac{8x}{27} + \frac{x^{3/2}}{8}, \quad \ldots(43)$$

and so on. Therefore, we get

$$u = u_0 + u_1 + u_2 + u_3 + \cdots$$
$$= 1 + x + x^{\frac{15}{10}} + 2 + \frac{2x}{3} + \frac{x^{3/2}}{2} + 4 + \frac{4x}{9} + \frac{x^{3/2}}{4} + 12 + \frac{8x}{27} + \frac{x^{3/2}}{8} + \cdots, \quad \ldots(44)$$

$$u = (1 + 2 + 4 + 12 + \cdots) + \left(x + \frac{2x}{3} + \frac{4x}{9} + \frac{8x}{27} + \cdots\right) + \left(x^{\frac{15}{10}} + \frac{x^{3/2}}{2} + \frac{x^{3/2}}{4} + \frac{x^{3/2}}{8} + \cdots\right), \quad \ldots(45)$$

$$u = x\left(\frac{1}{1-\frac{2}{3}}\right) + x^{\frac{3}{2}}\left(\frac{1}{1-\frac{1}{2}}\right) + \left(\frac{1}{1-2}\right) = 3x + 2x^{1.5} - 1, \quad \ldots(46)$$

Which is the exact solution of integral equation (39) [12, 17] obtain upon using of infinite geometric series in Eq. (20).

Example 5:

Considering the following linear WSVIE given in [17]:

$$u(x) = \frac{35}{33}x^{1.5} + \frac{5}{4}x + \frac{2}{3} + x^{-\frac{1}{3}}\int_0^x t^{-\frac{2}{3}}\, u(t)dt, \quad \ldots(47)$$

Where, $\mu = \frac{1}{3}$, $f(x) = \frac{35}{33}x^{1.5} + \frac{5}{4}x + \frac{2}{3}$, and the exact solution is $u(x) = \frac{7}{3}x^{1.5} + 5x - \frac{1}{3}$.
By using DJM, we get

$$u_0(x) = f(x) = \frac{35}{33}x^{1.5} + \frac{5}{4}x + \frac{2}{3}, \qquad \dots(48)$$

$$u_1(x) = N(u_0) = x^{-\frac{1}{3}}\int_0^x t^{-\frac{2}{3}} u_0(t)dt = 2 + \frac{15x}{16} + \frac{70x^{3/2}}{121}, \qquad \dots(49)$$

$$u_2(x) = x^{-\frac{1}{3}}\int_0^x t^{-\frac{2}{3}}(u_0 + u_1)(t)dt - u_1(x) = 6 + \frac{45x}{64} + \frac{420x^{3/2}}{1331}, \qquad \dots(50)$$

$$u_3(x) = x^{-\frac{1}{3}}\int_0^x t^{-\frac{2}{3}}(u_0 + u_1 + u_2)(t)dt - x^{-\frac{1}{3}}\int_0^x t^{-\frac{2}{3}}(u_0 + u_1)(t)dt$$
$$= 18 + \frac{135x}{256} + \frac{2520x^{3/2}}{14641}, \qquad \dots(51)$$

and so on. Thus, the approximate solution

$$u = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$u = \frac{35}{33}x^{1.5} + \frac{5}{4}x + \frac{2}{3} + 2 + \frac{15x}{16} + \frac{70x^{3/2}}{121} + 6 + \frac{45x}{64} + \frac{420x^{3/2}}{1331} + 18 + \frac{135x}{256} + \frac{2520x^{3/2}}{14641} + \cdots, \qquad \dots(52)$$

$$u = \left(\frac{35}{33}x^{1.5} + \frac{70x^{3/2}}{121} + \frac{420x^{3/2}}{1331} + \frac{2520x^{3/2}}{14641} + \cdots\right) + \left(\frac{5}{4}x + \frac{15x}{16} + \frac{45x}{64} + \frac{135x}{256} + \cdots\right) + \left(\frac{2}{3} + 2 + 6 + 18 + \cdots\right), \qquad \dots(53)$$

$$u = x^{\frac{3}{2}}\left(\frac{\frac{35}{33}}{1 - \frac{6}{11}}\right) + \left(\frac{\frac{5}{4}}{1 - \frac{3}{4}}\right)x + \left(\frac{\frac{2}{3}}{1 - 3}\right) = \frac{7}{3}x^{1.5} + 5x - \frac{1}{3}, \qquad \dots(54)$$

Which the exact solution for Eq. (47) [12, 17], obtain upon using the sum of infinite geometric series in Eq. (20).

Example 6:

Now, we consider the linear WSVIE [18]:

$$u(x) = x(1 - x) + \frac{16}{105}x^{7/2}(7 - 6x) - \int_0^x \frac{xt}{\sqrt{x - t}}u(t)dt, \qquad \dots(55)$$

Where, $\mu = \frac{1}{2}, f(x) = x(1 - x) + \frac{16}{105}x^{7/2}(7 - 6x)$, and the exact solution is $u(x) = x(1 - x)$. According to the DJM we achieve the following components

$$u_0(x) = f(x) = x - x^2 + \frac{16x^{7/2}}{15} - \frac{32x^{9/2}}{35}, \qquad \dots(56)$$

$$u_1(x) = N(u_0) = \int_0^x -\frac{xt}{(x - t)^{\frac{1}{2}}}u_0(t)dt = -\frac{16x^{7/2}}{15} + \frac{32x^{9/2}}{35} - \frac{21\pi x^6}{80} + \frac{33\pi x^7}{160}, \qquad \dots(57)$$

$$u_2(x) = \int_0^x -\frac{xt}{(x - t)^{\frac{1}{2}}}(u_0 + u_1)(t)dt - u_1(x)$$
$$= \frac{21\pi x^6}{80} - \frac{33\pi x^7}{160} + \frac{1792\pi x^{17/2}}{10725} - \frac{2048\pi x^{19/2}}{16575}, \qquad \dots(58)$$

$$u_3(x) = \int_0^x \frac{xt}{(x - t)^{\frac{1}{2}}}(u_0 + u_1 + u_2)(t)dt + \int_0^x \frac{xt}{(x - t)^{\frac{1}{2}}}(u_0 + u_1)(t)dt$$
$$= -\frac{1792\pi x^{17/2}}{10725} + \frac{2048\pi x^{19/2}}{16575} + \cdots, \qquad \dots(59)$$

and so on. Considering the first two components for u_0, u_1, u_2, we observe the appearance of the noise terms. By cancelling the identical terms with opposite signs gives the exact solutions. This called noise phenomena, for more details about necessary conditions see [1, 33]. Consequently, the approximate solutions are given by

$$u = x(1-x) + \left(\frac{16x^{7/2}}{15} - \frac{32x^{9/2}}{35} - \frac{16x^{7/2}}{15} + \frac{32x^{9/2}}{35}\right) + \left(-\frac{21\pi x^6}{80} + \frac{33\pi x^7}{160} + \frac{21\pi x^6}{80} - \frac{33\pi x^7}{160}\right) + \cdots, \quad \ldots (60)$$

Hence, the exact solution is $u\,(x) = x(1-x)$ for Eq. (55) [18].

4.2 Nonlinear WSVIE

In this subsection we will use the DJM to handle the nonlinear WSVIE. The noise terms phenomenon will be used wherever it is appropriate.

Case 1: $\mu > 1$:

We will discuss examples of the nonlinear WSVIE when $\mu > 1$, and will be solved by DJM to obtain the exact solution.

Example 7:

Consider the nonlinear WSVIE of second kind.

$$u(x) = \sqrt{x} - \frac{5}{11}x + x^{-\frac{12}{10}} \int_0^x t^{\frac{2}{10}} u^2(t)dt \ , \quad \ldots (61)$$

Where, $\mu = \frac{12}{10}, f(x) = \sqrt{x} - \frac{5}{11}x$, and the exact solution is $u\,(x) = \sqrt{x}$.

By using DJM, we get

$$u_0(x) = f(x) \ = \sqrt{x} - \frac{5}{11}x \ , \quad \ldots (62)$$

$$u_1\,(x) = N(u_0) = x^{-\frac{12}{10}} \int_0^x t^{\frac{2}{10}} u^2{}_0(t)dt \ = \frac{5x}{11} - \frac{100x^{3/2}}{297} + \frac{125x^2}{1936}, \quad \ldots (63)$$

$$u_2\,(x) = x^{-\frac{12}{10}} \int_0^x t^{\frac{2}{10}} (\,u_0 + u_1)^2(t)dt - u_1(x) = \frac{100x^{3/2}}{297} - \frac{14375x^2}{52272} + \frac{625x^{5/2}}{17908} + \cdots, \quad \ldots (64)$$

$$u_3\,(x) = x^{-\frac{12}{10}} \int_0^x t^{\frac{2}{10}} (\,u_0 + u_1 + u_2)^2(t)dt - x^{-\frac{12}{10}} \int_0^x t^{\frac{2}{10}} (\,u_0 + u_1)^2(t)dt$$

$$= -\frac{125x^2}{1936} + \frac{14375x^2}{52272} - \frac{71875x^{5/2}}{483516} + \cdots \quad , \quad \ldots (65)$$

$$u_4\,(x) = x^{-\frac{12}{10}} \int_0^x t^{\frac{2}{10}} (\,u_0 + u_1 + u_2 + u_3)^2(t)dt - x^{-\frac{12}{10}} \int_0^x t^{\frac{2}{10}} (\,u_0 + u_1 + u_2)^2(t)dt$$

$$= -\frac{625x^{5/2}}{17908} + \frac{71875x^{5/2}}{483516} + \cdots \quad , \quad \ldots (66)$$

In view of (66), the solution in a series form is given by

$$u = \sqrt{x} + \left(-\frac{5}{11}x + \frac{5x}{11} - \frac{100x^{3/2}}{297} + \frac{125x^2}{1936}\right) + \left(\frac{100x^{3/2}}{297} - \frac{14375x^2}{52272}\right) + \left(-\frac{125x^2}{1936} + \frac{14375x^2}{52272} + \cdots\right) + \cdots, \quad \ldots (67)$$

obtained upon using the decomposition (7). The solution in a closed form $u = \sqrt{x}$, is readily obtained by cancelling the all noise terms.

Example 8:

We consider the nonlinear WSVIE

$$u(x) = x - \frac{2x^3}{9} + x^{-\frac{15}{10}} \int_0^x t^{\frac{1}{2}} \ u^3(t)dt \ , \quad \ldots (68)$$

Let $\mu = \frac{15}{10}, f(x) = x - \frac{2x^3}{9}$, and the exact solution is $u\,(x) = x,$ using the recurrence relation defined in Eqs. (6) and (7), we obtain,

$$u_0(x) = f(x) \ = x - \frac{2x^3}{9} \ , \quad \ldots (69)$$

$$u_1(x) = N(u_0) = x^{-\frac{15}{10}}\int_0^x t^{\frac{1}{2}}\, u^3{}_0(t)dt \;= \frac{2x^3}{9} - \frac{4x^5}{39} + \frac{8x^7}{459} + \cdots, \qquad \ldots(70)$$

$$u_2(x) = x^{-\frac{15}{10}}\int_0^x t^{\frac{1}{2}}(u_0+u_1)^3(t)dt - u_1(x) = \frac{4x^5}{39} - \frac{320x^7}{5967} + \cdots, \qquad \ldots(71)$$

$$u_3(x) = x^{-\frac{15}{10}}\int_0^x t^{\frac{1}{2}}(u_0+u_1+u_2)^3(t)dt - x^{-\frac{15}{10}}\int_0^x t^{\frac{1}{2}}(u_0+u_1)^3(t)dt$$
$$= -\frac{8x^7}{459} + \frac{320x^7}{5967} + \cdots \quad, \qquad \ldots(72)$$

and so on. In view of (72), the solution in a series form is given by

$$u = \mathrm{x} + \left(-\frac{2x^3}{9} + \frac{2x^3}{9} - \frac{4x^5}{39} + \frac{8x^7}{459}\right) + \left(\frac{4x^5}{39} - \frac{320x^7}{5967} - \frac{8x^7}{459} + \frac{320x^7}{5967} + \cdots\right) + \cdots \qquad \ldots(73)$$

obtained upon using the decomposition (7). The solution in a closed form $u = x$, is readily obtained by cancelling the all noise terms.

Case 2: 0 < μ ≤ 1:

We will discuss examples of the nonlinear WSVIE when $0 < \mu \leq 1$, and will be solved by DJM to obtain the exact solution.

Example 9:

Consider the nonlinear WSVIE [34]

$$u(x) = x^{1/2} + \frac{3\pi}{8}x^2 + \int_0^x -\frac{1}{(x-t)^{\frac{1}{2}}}u^3(t)dt \;, \qquad \ldots(74)$$

Let $\mu = \frac{1}{2}$, $f(x) = x^{1/2} + \frac{3\pi}{8}x^2$, and the exact solution is $u(x) = \sqrt{x}$. Applying the DJM, we obtain the following components:

$$u_0(x) = f(x) \;= x^{1/2} + \frac{3\pi}{8}x^2 \;, \qquad \ldots(75)$$

$$u_1(x) = N(u_0) = \int_0^x -\frac{1}{(x-t)^{\frac{1}{2}}}u_0^3(t)dt$$
$$= -\frac{3\pi x^2}{8} - \frac{36}{35}\pi x^{7/2} - \frac{1701\pi^3x^5}{16384} - \frac{36\pi^3x^{13/2}}{1001}, \qquad \ldots(76)$$

$$u_2(x) = \int_0^x -\frac{1}{(x-t)^{\frac{1}{2}}}(u_0+u_1)^3(t)dt - u_1(x)$$
$$= \frac{36}{35}\pi x^{7/2} + \frac{243\pi^2x^5}{320} + \frac{1701\pi^3x^5}{16384} + \cdots \;, \qquad \ldots(77)$$

$$u_3(x) = u_3(x) = \int_0^x -\frac{1}{(x-t)^{\frac{1}{2}}}(u_0+u_1+u_2)^3(t)dt - \int_0^x -\frac{1}{(x-t)^{\frac{1}{2}}}(u_0+u_1)^3(t)dt$$
$$= -\frac{243}{320}\pi^2x^5 \;+\cdots \quad, \qquad \ldots(78)$$

$$u_4(x) = \sqrt{x} + \left(\frac{3\pi}{8}x^2 \;- \frac{3\pi x^2}{8} - \frac{36}{35}\pi x^{7/2} - \frac{1701\pi^3x^5}{16384} + \cdots\right)$$
$$= + \left(\frac{36}{35}\pi x^{7/2} + \frac{243\pi^2x^5}{320} + \frac{1701\pi^3x^5}{16384} + \cdots\right) + \left(-\frac{243}{320}\pi^2x^5 \;+\cdots\right) + \cdots \quad, \qquad \ldots(79)$$

which is the exact solution of integral equation (74) [34].

Example 10:

Consider the following nonlinear WSVIE [34]

$$u(x) = x^{1/2} - \frac{16}{15}x^{5/2} + \int_0^x \frac{u^4(t)}{(x-t)^{\frac{1}{2}}}dt, \qquad \ldots(80)$$

Let $\mu = \frac{1}{2}$, $f(x) = x^{1/2} - \frac{16}{15}x^{5/2}$, and the exact solution is $u(x) = \sqrt{x}$. Using the recurrence relation defined in Eqs. (6) and (7), we obtain,

$$u_0(x) = f(x) = x^{1/2} - \frac{16}{15}x^{5/2} \;, \qquad \ldots(81)$$

$$u_1(x) = N(u_0) = \int_0^x \frac{u_0^4(t)}{(x-t)^{\frac{1}{2}}}dt = \frac{16x^{5/2}}{15} - \frac{16384x^{9/2}}{4725} + \frac{1048576x^{13/2}}{225225} + \cdots , \quad \ldots(82)$$

$$u_2(x) = \int_0^x \frac{(u_0+u_1)^4(t)}{(x-t)^{\frac{1}{2}}}dt - u_1(x) = \frac{16384x^{9/2}}{4725} - \frac{200278016x^{13/2}}{14189175} + \cdots , \quad \ldots(83)$$

$$u_3(x) = \int_0^x \frac{(u_0+u_1+u_2)^4(t)}{(x-t)^{\frac{1}{2}}}dt - \int_0^x \frac{(u_0+u_1)^4(t)}{(x-t)^{\frac{1}{2}}}dt$$
$$= -\frac{1048576x^{13/2}}{225225} + \frac{200278016x^{13/2}}{14189175} + \cdots \quad , \quad \ldots(84)$$

and so on. In view of (82), the solution in a series form is given by

$$u = \sqrt{x} + \left(-\frac{16}{15}x^{5/2} + \frac{16x^{5/2}}{15} - \frac{16384x^{9/2}}{4725} + \frac{1048576x^{13/2}}{225225} + \cdots\right)$$
$$+ \left(\frac{16384x^{9/2}}{4725} - \frac{200278016x^{13/2}}{14189175} - \frac{1048576x^{13/2}}{225225} + \frac{200278016x^{13/2}}{14189175} \ldots\right) + \cdots, \quad \ldots(85)$$

obtained upon using the decomposition (7). The solution in a closed form $u(x) = \sqrt{x}$. is readily obtained by cancelling the all noise terms.

V. Conclusion

In this paper, an efficient iterative method namely (DJM) is implemented to obtain the exact solutions for solving linear and nonlinear WSVIE in two cases: case 1: $\mu > 1$ and case 2: $0 < \mu \leq 1$. In this method the solution is obtained in the series form that converges to the exact solution with easily computed components. The DJM is used directly without required or restricted assumptions for the nonlinear terms and the main advantage of the method is its simplicity in solutions also this method does not require complicated calculations. The method gives rapid convergent and can be easily comprehended with only a basic knowledge of Calculus. It is economical in terms of computer power/memory and does not involve tedious calculations. Moreover, by solving some examples, it is seems that the DJM appears to be very accurate to employ with reliable results. The software used for the calculations in this study was MATHEMATICA® 10.0.

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