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Prime Ring, Semiprime Ring and Their Connection to Quotient Ring

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Abstract: *A ring **R** is called (left) Artinian if it is Artinian as a left module over itself. In this paper, we have discussed about the prime ring, semi-prime ring and their connection to quotient ring. Mainly, we focused on " A right quotient ring which is a semi-simple Artinian ring if and only if the ring is a semi-prime ring, has a finite rank and has a maximum condition on right annihilators".*
Keywords: *Quotient ring, Prime ring, prime field, Artinian ring, Annihilator, Char(R).*

I. Introduction

We have described some definitions and in discussion some properties of prime ring, semi-prime ring, quotient ring and characteristic of a ring have been highlighted. Finally, due to Goldie's theorem we have prove that " A right quotient ring which is a semi-simple Artinian ring if and only if the ring is a semi-prime ring, has a finite rank and has a maximum condition on right annihilators".

II. Discussion

An ideal P of R is prime, in case, given right ideals A, B,then $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. The product RB is an ideal of R; hence,
$AB = (AR)B = A(RB) \subseteq P \Leftrightarrow R(AB) = (RA)(RB) \subseteq P$.

It follows that P is prime if and only if $\forall$ ideals A', B' of R $(\dot{A}'B' \subseteq P \Leftrightarrow A' \subseteq P$ or $B' \subseteq P)$.

Thus, a maximal ideal must be prime since R is the only ideal containing it. By symmetry, an ideal P is prime if and only if the defining statement is valid with "left" substituting "right".
A ring R is prime provided that O is a prime ideal ; R is semi-prime if O is only nilpotent ideal. Thus, every prime ring is semi-prime.

Every semi simple ring R is semi-prime: If I is a right ideal, then I is a summand of R, so $I = eR$ for some idempotent e, hence $I^n = I$ for any $n \geq 1$. Thus R is semi-prime.
Depending on the characteristic of a ring its prime subring will be finite or infinite.
One important property of semi-prime ring is that its simple right ideals are generated by idempotent element.

2.1 Char(R):

Given a ring R (commutative or not, with or without unity), by the characteristic of R, denoted by $char(R)$, we mean the least positive integer n such that $na = 0$, $\forall\ a \in R$ if such an n exists ; otherwise, it is defined to be 0.

If a ring R has 1, then we have

1) $char(R) = 0$ if and only if the additive order of 1 in the abelian group $(R,+)$ is infinite.
2) $char(R) = n \neq 0$ if and only if the additive order of 1 in $(R,+)$ is finite and is equal to n.

Let R be a ring with 1. Let $p = \{n|n \in Z\}$ be the smallest sub ring of R containing 1, called the prime sub-ring of R. Then we have the following

Suppose R is a ring with 1 such that non-units in R form a subgroup of $(R,+)$, then $char(R)$ is either 0 or else a power of a prime.

Any division ring D (in particular, a field) contains either Q or Z/pZ as the smallest subfield contained in the centre which is called centre subfield of D according as the characteristics of D is 0 or a prime p.

Let S be a sub-ring of ring R. Then $char(S) \le char(R)$ and equality holds if both R and S have the same unity. (However, if S has unity but different from that of R, then equality may or may not hold.)

2.2 Examples:

1) $Char(Z) = Char(Q) = char(R) = char(C) = char(H) = 0$.
2) $Char(M_n(R)) = char(R)$.
3) $Char(R) = char(R[x]) = char(R[[x]]) = char(R\langle x \rangle)$.
4) $Char(Z_n) = n$.
5)

2.3 Prime field:

The smallest (central) subfield of a division ring D is called the prime field of D (and it is Q or Z/pZ according as $charD = 0$ or p).

For any ring R with 1, R contains Z or Z/nZ as the smallest central sub-ring containing 1 according as $charR$ is 0 or n and it is called the prime sub-ring of R. In particular,

- If R is an integral domain with 1, then R contains Z or Z/pZ as its prime sub-ring according as $charR$ is 0 or a prime p and
- If R is a commutative integral domain with 1, then $Q(R)$ contains Q or Z/pZ as its prime field according as $charR$ is 0 or p.

If D and D' are division rings such that there is a non-zero homomorphism of rings $f : D \to D'$, then $charD = charD'$ and f is identity on the prime field of D. (This is trivial because $f(1) = 1$ and hence f is identity on Z or Z/pZ, etc.)

If I is a simple ideal in a ring R, then either $I^2 = 0$ or I is generated by an idempotent.

If R is a semi prime ring, every simple right ideal is generated by an idempotent.

A prime ring is a semi-prime ring. A commutative prime ring is an integral domain. An Artinian semi-prime ring is semi simple and if prime is a simple ring which is a full matrix ring over a division ring.

The following theorem is due to Goldie. It has been proved for prime rings under two sided conditions.

III. Artinian Ring

3.1 Theorem:

A ring R has right quotient ring Q which is a semi-simple Artinian ring if and only if

1. R is a semi-prime ring,
2. R has finite right rank,
3. R has maximum condition on right annihilators.

The third condition needs explanation. Let S be a subset of R. We denote $r(S) = (x \in R; xS = 0)$.

Then $r(S)$ is a right ideal, the right annihilator of S. Left annihilators are similarly defined as $l(S) = (x \in R; xS = 0)$.

An interesting ideal in any ring R is the set $Z(R) = (Z \in R; r(Z)$ is an essential right ideal). $Z(R)$ is a two sided ideal of R. In the presence of the maximum condition on right annihilators $Z(R)$ becomes a nil ideal. R. E. Johnson has given this concept.

3.2 Lemma:

Let R be a ring satisfying conditions (1) and (3). Then a nil right (left) ideal of R is zero.

Proof: As aR is nil if and only if Ra is nil, we consider $Ra \neq 0$. The set of right annihilators $r(za)$, z ranging over R with $za \neq 0$, has maximal elements. Let $r(b)$ be maximal, $b = ta$. Suppose that $(yb)^k = 0$, $(yb)^{k-1} \neq 0$, where $y \in R$, and $k \geq 1$. Then $r(b) = r((yb)^{k-1})$ by the maximality and hence $byb = 0$. Thus $bRb = 0$ and $(Rb)^2 = 0$, $b = 0$, since R has no nilpotent left ideals. This contradiction shows that $Ra = 0$ and the lemma follows.

3.3 Lemma:

Let R be any ring which satisfies conditions (2) and (3). For each $a \in R$ there exists $n > 0$ such that $a^n R + r(a^n)$ is an essential right ideal.

Proof: Because of (3) there is an $n \geq 1$ such that $r(a^n) = r(a^{n+1})$. Then $a^n R \cap r(a^n) = 0$. Let I be a right ideal and suppose that $I \cap (a^n R + r(a^n)) = 0$. The sum $I + a^n I + a^{2n} I + \cdots$ is direct and, because R has finite rank, we conclude that $I = 0$. Lemma1 is due to Utumi as regards proof and Lemma 2 uses an idea of Lesieur-Crosot.

Proof of The Theorem 3.1

Under the stated conditions we prove that an essential right ideal E contains a regular element. E is not a nil ideal; it has an element $a_1 \neq 0$ with $r(a_1) = r(a_1{}^2)$. Either $r(a_1) \cap E = 0$ or $r(a_1) \cap E \neq 0$. In the latter case, choose $a_2 \in r(a_1) \cap E$ with $a_2 \neq 0$ and $r(a_2) = r(a_2{}^2)$. If $r(a_1) \cap r(a_2) \cap E \neq 0$, then the process continues. At the general stage we have a direct sum $a_1 R \oplus \cdots \oplus a_k R \oplus (r(a_1) \cap \cdots \cap r(a_k) \cap E)$, where $a_k \in (r(a_1) \cap \cdots \cap r(a_{k-1}) \cap E)$ and $a_k \neq 0$, $r(a_k) = r(a_k{}^2)$. The process has to stop, because R has finite rank; let this happen at the k-th stage.

Then $r(a_1) \cap \cdots \cap r(a_k) \cap E = 0 = r(a_1) \cap \cdots \cap r(a_k)$

and hence $r(a_1{}^2 + \cdots + a_k{}^2) = (r(a_1) \cap \cdots \cap r(a_k)) = 0$.

Let Z be the singular ideal of R and $z \in Z$. Then $z^n R \oplus r(z^n)$ is an essential right ideal for some $n > 0$ and $r(z^n)$ is also essential. Hence $z^n R = 0$, Z is a nil ideal and hence is zero. Set $c = a_1{}^2 + \cdots + a_k{}^2 \in E$; as $r(c) = 0$ we deduce that cR is essential by Lemma 2. Hence $l(c) \subset Z$, so that $l(c) = 0$ and c is a regular element. This establishes the existence of regular elements in R.

Suppose that $a, d \in R$ with d regular and set $E = (x \in R;\ ax \in dR)$. Then dR is essential and hence so is E, so that E contains a regular element d_1. The right Ore condition is satisfied and R has a right quotient ring Q.

Next suppose that F is an essential right ideal in Q, then $F \cap R$ is essential in R. Now $F \cap R$ has a regular element, which is a unit in Q, and hence $F = Q$. Let J be a right ideal and K a right ideal of Q such that $J \cap K = 0$ and $J \oplus K$ is essential (use Zorn's Lemma); then $J \oplus K = Q$. Thus the module Q_Q is semi-simple and Q is a semi-simple ring.

Conversely, let R have a semi-simple right quotient ring Q. Then a right ideal E of R is essential if and only if $EQ = Q$. To see this, suppose I is a non-zero right ideal of Q, then $I \cap R \neq 0$ and $I \cap R \cap E \neq 0$, taking E to be essential in R. Hence $I \cap EQ \neq 0$, which means that EQ is essential in Q; then $EQ = Q$ as Q_Q is a direct sum of simple modules. On the other hand, when $EQ = Q$ is given and I is a non-zero right ideal of R then $IQ \cap EQ \neq 0$, trivially and hence $I \cap E \neq 0$, so that E is an essential right ideal of R. These conditions are equivalent to saying that E has a regular element, because $y \in EQ$ and $1 = ec^{-1}$, $e, c \in R$, and regular. On the other hand, when E has a regular element, then $EQ = Q$ and E is essential in R. We now conclude that R is a semi-prime ring, because if N is a nilpotent ideal of R, $l(N)$ is essential as right ideal and has a regular element. Thus $N = 0$. Let $S = (\sum I_\alpha; \alpha \in A)$ be a direct sum of non-zero right ideals of R which is essential as a right ideal. S has a regular element c, expressible as a finite sum $c = x_1 + \cdots + x_n; x_i \in I_{\alpha_i}$.

Now cR is essential and lies in $I_{\varepsilon_1} + \cdots + I_{\alpha_n}$; it follows that A has only the indices $\alpha_1, \cdots, \alpha_n$, and R has finite right rank.

Finally, the maximum condition holds for right annihilators, because $r_Q(S) \cap R = r_R(S)$ for any non-empty subset S of R, the subscripts denoting the ring in which the annihilator is taken.

The following corollary is evident.

Corollary:

A prime ring R has a simple Artinian ring Q as its right quotient ring if and only if conditions (2) and (3) hold.

This follows, since Q is a full ring of matrices over a division ring ; $Q = D_n$, say.

Corollary:

A prime ring R with conditions (2) and (3) contains a prime ring R' which also has Q as its right quotient ring and R' is a full matrix rings C_n where C has the division ring D as its right quotient ring. This corollary is due to C. Faith and Y. Utumi.

IV. Conclusion

From the above discussion, it conclude that lemma 3.2 shows that if a semi-simple right quotient Artinian ring R has finite right rank and has maximum condition on right annihilators then a nil right (left) ideal of R is zero.

Lemma 3.3 ensures that if any ring R which satisfies condition (2) and (3) of theorem 3.1. For each $a \in R$ there exists $n > 0$ such that $a^n R \cap r(a^n)$ is an essential right ideal.

Lemma 3.2, 3.3 together with Goldie's theorem establishes that, a ring R has right quotient ring Q which is a semi-simple Artinian ring if and only if

1. R is a semi-prime ring,
2. R has finite right rank,
3. R has maximum condition on right annihilators.

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