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On Statistical Convergence Of Double Sequences And Statistical Monotonicity

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Abstract: *Following the introduction of the concept of statistical monotonicity and upper (or lower) peak point of real valued single sequences by Kaya et al in 2013, we shall in this paper investigate properties of statistically convergent double sequences, introduce definitions of statistical monotonicity and lower (upper) peak points of real valued double sequences. And establish the relationships between the statistical convergence of double sequences and these notions. Finally, we generalised statistical monotonicity using an RH −regular doubly infinite matrix transformation.*

Keywords: *Statistical convergence of double sequences, lower to upper peak point and statistical monotone.*
Mathematics subject classification: *Primary 40F05, 40J05, 40G05.*

I. Introduction

Pringsheim (1900) introduced the concept of convergence for double sequences. Also Robison (1926) and Hamilton (1936, 1938a, 1938b, 1939) studied the four dimensional matrix transformation $(Ax)_{mn}$ $=\sum_{k,l=0,0}^{\infty,\infty} a_{m,n\ k,l}\ x_{kl}$ extensively. Using the above concept of Patterson (1999 and 2000) formulated some analogous of fundamental theorems of summability and double sequence core theorem. Later on many researches have been done in the field of statistical convergence of double sequence [see for example Teripathy, (2003)], Mursaleen & Edely, (2003), Siddiqui et al (2012), Brono and siddiqui (2013)] and many others. Combining this studies and the concept of statistical monotonicity and statistical convergence as introduced in Kaya et al (2013); we present analogous extension of the various concepts of Kaya et al (2013) to double sequences theorems.

Definition1.1 Pringsheim (1900): A double sequence $x = (x_{jk})$ is said to be Pringssheim's convergent (or P-convergent) if for given $\varepsilon > 0$ there exists an integer ℕ such that $|x_{jk} - \ell| < \varepsilon$ whenever $j, k > N$. In this case ℓ is called the Pringsheim limit of $x = (x_{jk})$ and it is written as $P - \lim x = \ell$.

Definition 1.2 [Mursaleen and Edely (2003)]: Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K_{m,n} = \{(j,k): j \le m, k \le n\}$. then the two-dimensional analogue of natural density can be defined as follows:

In case the sequence $K(m,n)/mn$ has a limit in the pringsheim's sense, then we say that K has a double natural density and is defined as

$$P - \lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(\mathrm{K}).$$

Example 1.1: Let $K = \{(i^2, j^2): i, j \in \mathbb{N}\}$. Then

$$\delta_2(\mathrm{K}) = P - \lim_{m,n} \frac{K(m,n)}{mn} \le P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

i.e. the set K has doubled natural density zero, while the set $\{(i,\ 2j): i, j \in \mathbb{N}\}$ has natural density $\frac{1}{2}$.

Definition 1.3: A real double sequence $x = (x_{jk})$ is said to be statistically convergent to the number ℓ if for each $\varepsilon > 0$, the set $\{(j,k), j \le m, k \le n: |x_{jk} - \ell| \ge \varepsilon\}$

Has natural density zero. In this case we write $St_2 - \lim_{j,k} x_{jk} = \ell$ and we denote the set of all statistically convergent double sequences by St_2. Deeply connected with this definition is the concept of strongly Cesáro summability for double sequences[see Mursaleen & Edely (2003)]

The following definitions of Cesáro summable double sequences is taken from [Moricz (1994)]

Definition1.4: Let $x = (x_{jk})$ *be a double* sequence. It is said to be Cesáro summable to ℓ *if*

$$\lim_{m,n} \frac{1}{mn} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = \ell.$$

We denote the space of all Cesáro summable double sequences by $(C, 1.1)$.

Similarly we can define the following as in case of single sequence

Definition1.5: Let $x = (x_{jk})$ be a double sequence and α be a positive real number. Then the double sequence x is said to be strongly $\alpha -$ Cesáro summable to ℓ if

$$\lim_{m,n} \frac{1}{mn} \sum_{j=1}^{n} \sum_{k=1}^{m} \left|x_{jk} - \ell\right|^{\alpha} = 0.$$

We denote the space of all strongly $\alpha -$ Cesáro summable double sequence by ω^2.

Remark 1.1: The space of all complex valued sequences $x = (x_n)$ will be denoted by $\mathbb{C}^{\mathbb{N}}$. In many circumstances we refer to $\mathbb{C}^{\mathbb{N}}$ as the space of arithmetical functions $f: \mathbb{N} \to \mathbb{C}$, especially, when f reflects the multiplicative structure of $\mathbb{N}$.

Remark 1.2: An Orlicz function f is a mapping $f: [0, \infty] \to [0, \infty]$ such that it is continuous non-decreasing and convex with $f(0) = 0, f(x) > 0,$ for > 0 and $f(x) \to \infty$ as $x \to \infty$. An Orlicz function f is said to satisfyΔ_2 $-$condition if there exists a consant $k > 0$ such that$f(2u) \le kf(u)$ for all values of $u \ge 0$.

Remark 1.3: Analogously, let a double Orlicz function f_D be a mapping $f_D: [0, \infty) \times [0, \infty) \to [0, \infty)$ such that it is continuous non-decreasing and convex with $f_D(0) = 0, f_D(x_{jk}) > 0,$ for $x_{jk} > 0, j, k = 1,2, \ldots .$ and $f_D(x_{jk}) \to \infty$ as $x_{jk} \to \infty$.

II. Some results about statistical convergence of double sequence

Define the function $p: \mathbb{C}^{\mathbb{N}} \times \mathbb{C}^{\mathbb{N}} \to [0, \infty)$ for all $x_{jk}, y_{ts} \in \mathbb{C}^{\mathbb{N}}$ as follows,

$$p(x, y) = \lim_{m,n} \frac{1}{mn} \sum_{jk \le mn} \varphi(|x_{jk} - y_{jk}|)$$

$$where\ \varphi: [0, \infty] \to [0, \infty)$$

$$\varphi(t) = \begin{cases} t, & if\ t \le 1, \\ 1, & 0therwise \end{cases}$$

It is clear that p is a semi$-$metric on $\mathbb{C}^{\mathbb{N}}$. Now we have

Theorem 2.1: The sequence $x = (x_{jk})$ is statistically convergent to ℓ if and only if $p(x, y) = 0$ where $y = (y_{jk})$ and $y_{jk} = \ell$ for all $j, k \in \mathbb{N}$**.**

Proof : Let us assume $p(x, y) = 0$ where $y_{jk} = \ell$ for all $j, k \in \mathbb{N}$. Then, if $\varepsilon > 0$

$$\lim_{m,n} \sup \frac{1}{mn} \sum_{\substack{j,k \le m,n \\ |x_{jk} - \ell|}} 1 \le max\left\{1, \frac{1}{\varepsilon}\right\} \lim_{m,n} \frac{1}{mn} \sum_{j,k \le m,n} \varphi(|x_{jk} - \ell|) = max\left\{1, \frac{1}{\varepsilon}\right\} p(x_{jk}, y_{jk}) = 0$$

and $x_{jk} \to \ell(S)$.

Now, assume that x is statistically convergent to ℓ. Then, for any $\varepsilon > 0$,

$$\frac{1}{mn} \sum_{j,k \le m,n} \varphi(|x_{jk} - \ell|) = \frac{1}{mn} \sum_{\substack{j,k \le m,n \\ |x_{jk} - \ell| < \varepsilon}} \varphi(|x_{jk} - \ell|) + \frac{1}{mn} \sum_{\substack{k \le n \\ |x_{jk} - \ell| \ge \varepsilon}} \varphi(|x_{jk} - \ell|) \le \varepsilon + \frac{1}{mn} \sum_{\substack{k \le n \\ |x_{jk} - \ell| \ge \varepsilon}} 1$$

which implies immediately

$p(x, y) \le \varepsilon$ for any $\varepsilon > 0$

Where $y = (y_{jk})$ and $y_{jk} = \ell$ $(jk \in \mathbb{N})$. Hence the proof.

The following can be seen from the above proof.

Corollary 2.1: If x_{jk} is strongly Cesáro summable to ℓ then x_{jk} is statistically convergent to ℓ.

Remark 2.1: The inverse of corollary 2.1 is not true in general. Consider the sequence x_{jk}:$\mathbb{N} \times \mathbb{N} \to \mathbb{C}$ as

$$x_{jk} \coloneqq \begin{cases} \sqrt{jk} & j, k = n^2, \quad n = 1,2, \ldots, \\ o, & otherwise. \end{cases}$$

On the other hand, we have

Corollary 2.2: If $x = (x_{jk})$ is a bounded sequence and statistically convergent to ℓ, then x_{jk} is strongly Cesáro summable to ℓ. Note a convergent double sequence need not be bounded.

The next theorem is well-known [see J.A. Fridy (1985)]

Theorem 2.2: A sequence x is statistically convergent to ℓ if and only if there exists $H \subset \mathbb{N}$ with $\delta(H) = 1$ such that x is convergent to ℓ in H, i.e.

$$\lim_{\substack{n \to \infty \\ n \in H}} x_n$$

Analogously, we extend this result to double sequences as follow:

Theorem 2.2.1: A double sequence $x = (x_{jk})$ is statistically convergent to ℓ if and only if there exists $H \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(H) = 1$ such that x is convergent to ℓ in H. i.e

$$\lim_{\substack{jk\to\infty \\ jk\in H}} x_{jk}$$

Proof: Assume that x_{jk} is statistically convergent to ℓ. There is $z_i \in \mathbb{N}$ such that

$$\frac{1}{mn} \sum_{\substack{j,k\leq m,n \\ 1/2^i<1/2^i}} 1 \leq \frac{1}{2^i}$$

Is satisfied for all $m,n \geq z_i$. Denote the set

$$H_i = \left\{jk \in \mathbb{N}: \frac{1}{2^i} \leq |x_{jk} - \ell| < \frac{1}{2^i}\right\}.$$

Then

$$\frac{1}{mn} \sum_{\substack{j,k\in\mathbb{N} \\ \frac{1}{2^i}<|x_{jk}-\ell|<\frac{1}{2^{i-1}}\ and\ j,k\in\mathbb{N}/H_i}} 1 < \frac{1}{2^i}$$

holds for all $jk \in \mathbb{N}$. If we consider the set $H = \bigcup_{i=1}^{\infty} H_i \cup \{jk : x_{jk} = \ell\}$ then $|x_{jk} - \ell| \geq \varepsilon$ hold only to finitely many $jk \in H$. This means that x is convergent to ℓ in the usual case. Now, let show that $\delta_2(\mathbb{N}/H) = 0$. Let $\varepsilon > 0$ be given and choose an arbitrary $r \in \mathbb{N}$ such that

$$\sum_{i=r+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}$$

holds. For r, there exists a $l_r \in \mathbb{N}$ such that

$$\frac{1}{mn}\sum_{\substack{j,k\leq m,n \\ \frac{1}{2^i}<|x_{jk}-\ell|<\frac{1}{2^{i-1}}\ and\ jk\in\mathbb{N}/H}} 1 < \frac{1}{r+1}.\frac{\varepsilon}{7} \quad \text{and} \quad \frac{1}{mn}\sum_{\substack{j,k\leq m,n \\ |x_{jk}-\ell|>1\ and\ jk\in\mathbb{N}/H}} 1 < \frac{1}{r+1}.\frac{\varepsilon}{4}$$

for all $m,n > l_r$ and $i = \{1,2,\dots,r\}$. Therefore,

$$\frac{1}{mn}\sum_{\substack{j,k\leq n \\ j,k\in\mathbb{N}/H}} 1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

Hold for all $m,n \geq l_r$.

The inverse of theorem is easily obtained.

III. Some results for double Orlicz functions

The following are some results for Orlicz functions. With $M(f_D)$ we denote the mean-value of the Orlicz function f_D, if the limit

$$M(f_D) = \lim_{m,n\to\infty} \frac{1}{mn} \sum_{j,k\leq n} f_D(j,k)$$

Theorem 3.1: Assume that $f: \mathbb{N} \to \mathbb{C}$ is bounded and statistically convergent to ℓ and $H \subset \mathbb{N} \times \mathbb{N}$ is an arbitrary set which possesses a double asymptotic density $\delta_2(H)$. Then, $M(1_H.f_D)$ exists and equal $\ell.\delta_2(H)$.

Proof: Consider the following inequality:

$$\left|\frac{1}{mn}\sum_{\substack{j,k\leq n \\ j,k\in\mathbb{N}}} f(jk) - \frac{1}{mn}\sum_{\substack{j,k\leq n \\ j,k\in\mathbb{N}}} \ell\right| \leq \frac{1}{mn} \sum_{\substack{j,k\leq n \\ k\in H, |f(jk)-\ell|<\varepsilon}} |f(jk) - \ell| + \frac{1}{mn}\sum_{\substack{j,k\leq n \\ jk\in H}} |f(jk) - \ell|.$$

Theorem 3.2: If a double Orlicz f_D is bounded and statistically convergent to $\ell \neq 0$, $f_D \equiv 1$.

Proof: Let $p_0 \in P, P$ is the of primes $j_0, k_0 \in \mathbb{N}$ and let

$H = \{n \in \mathbb{N}: p_0^{j_0k_0} \| n\}$,

Be the set of all elements of $\mathbb{N}$ divisible exactly by $p_0^{k_0}$, i.e. n can be written in the form $n = p_0^{k_0} z$ where $p_0 \nmid z$. It is clear from Theorem 3.1 that

$M(1_H.f_D) = \ell\delta_2(H) = \ell \frac{1}{p_0^{j_0k_0}}\left(1 - \frac{1}{p_0}\right)$

holds. Since f_D is multiplicative, we have

$f_D(jk) = f(p_0^{j_0k_0}).f_D\left(\frac{jk}{p_0^{j_0k_0}}\right)$ for $j,k \in H$.

Therefore,

$$M(1_H.f_D) = \lim_{m,n\to\infty} \frac{1}{mn} \sum_{\substack{jk\leq n \\ j,k\in H}} f_D(jk)$$

$$= \lim_{m,n\to\infty} f_D\left(p_0^{j_0k_0}\right) \sum_{\substack{z\leq \frac{mn}{p_0^{j_0k_0}} \\ p_0\neq z}} f_0(z) = \lim_{m,n\to\infty} f_D\left(p_0^{j_0k_0}\right) \frac{1}{p_0^{j_0k_0}} \cdot \frac{1}{\frac{mn}{p_0^{j_0k_0}}} \sum_{\substack{z\leq \frac{mn}{p_0^{j_0k_0}} \\ p_0\nmid z}} f_0(z)$$

$$= f_D\left(p_0^{j_0k_0}\right) \frac{1}{p_0^{j_0k_0}} . \ell . \left(1 - \frac{1}{p_0}\right)$$

This implies $f\left(p_0^{j_0k_0}\right)$. Since p_0 is a prime $j_o k_0 \in \mathbb{N}$ have been chosen arbitrarily it follows $f_D = 1(and\ \ell = 1$.

Remark 3.1: If f_D is bounded and statistically convergent to ℓ, then f_D is Cesáro summable $(C, 1.1)$ (for every $p > 0$). And we extend Theorem 3 of Indlekofer(1986) as follows:

Proposition 3.1: Let f_D be an Orlicz function and $\alpha > 0$. Then the following results hold; (i) if f_D is Cesáro summable (C,1.1) for $\ell \neq 0$, then $\ell = 1$ and $f_D(jk) = 1$ for all $j, k \in \mathbb{N}$.

(ii) f_D is Cesáro summable (C,1.1) for $\ell = 0$ if and only if $|f_D|^\alpha \in \ell^*$ and one of the series

$$\sum_{\substack{p_{st} \\ |f_D|p_{st}||\leq\frac{1}{2}}} \frac{(|f_D|p_{st}||-1)^2}{p_{st}}, \sum_{\substack{p_{st} \\ |f_D|p_{st}|-1|>\frac{1}{2}}} \frac{(|f_D|p_{st}||-1)^\alpha}{p_{st}}$$

Diverges or

$\sum_{p_{st}\leq x_{jk}} \frac{|f_D(p_{st})|-1}{p_{st}} \to -\infty$ as $x_{jk} \to \infty$. $j, k, s, t = 1,2, \ldots$,I

In other words (i) above is the as theorem 3.2.

IV. Statistical monotonicity for double sequences and some related results

Here we shall consider only real-valued double sequences and introduce the concept of statistical monotonicity.

Definition 4.1: (statistical monotone increasing (or decreasing) sequence)

A sequence $x = (x_{jk})$ is statistical monotone increasing (decreasing) if there exists a subset $H \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(H) = 1$ such that the sequences $x = (x_{jk})$ is monotone increasing (or decreasing) on H.

A sequence $x = (x_{jk})$ is statistical monotone if it is statistical monotone increasing or statistical monotone decreasing.

In the following we list some (obvious) properties of statistical monotone sequences.

(i) If the sequence $x = (x_{jk})$ is bounded and statistical monotone then it is statistically convergent.

(ii) If $x = (x_{jk})$ is statistical monotone increasing or statistical monotone decreasing then

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\{jk: j \leq m, k \leq n: x_{j+1,k+1} < x_{jk}\}\right| = 0 \qquad (1)$$

or

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\{jk: j \leq m, k \leq n: x_{j+1,k+1} > x_{jk}\}\right| = 0 \qquad (2)$$

respectively. The inverse of these assertions is not necessarily true because of the following example:

Define $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} 1, & if 2^i \leq j < 2^{i+1} - 1,\ for\ even\ i \\ 0, & otherwise. \end{cases}$$

Then the relation (1) and (2) hold but $x = (x_{jk})$ is said to be statistical bounded if there exists a number $M > 0$ such that

$$\delta(\{jk \in \mathbb{N}: |x_{jk}| > M\}) = 0.$$

Let $\{n_j^i\}$ and $\{k_j^i\}$ be a strictly increasing double index sequences of positive natural numbers and $x = (x_{jk})$, define $x' = \left(x_{j_{ik_i}}\right)$ and $K_i := \{j_i k_i: i \in \mathbb{N}\}$

For more on construction of subsequences of double sequences [see Patterson(1999) & (2000)].

Definition 4.3: (Dense subsequence) the subsequence $x' = \left(x_{j_{ik_i}}\right)$ of $x = (x_{jk})$ is called a dense subsequence, if $\delta(K_{x'}) = 1$.

(iii) Every dense subsequence of a monotone double sequence is statistical monotone.

(iv) The statistical monotone double sequence $x = (x_{jk})$ is statistical convergent if and only if $x = (x_{jk})$ is statistical bounded.

Definition 4.4: The double sequence $x = (x_{jk})$ and $y = (y_{jk})$ are called statistical equivalent if there is a subset M of $\mathbb{N}$ with $\delta(M) = 1$ such that $x_{jk} = y_{jk}$ for each $j, k \in M.$ It is denoted by $x_{jk} \asymp y_{jk}$.

(v) Let $x = (x_{jk})$ and $y = (y_{jk})$ be statistical equivalent. Then $x = (x_{jk})$ statistical monotone if and only if $y = (y_{jk})$ is statistical monotone.

V. Peak points for double sequences and some related results

In this section upper and lower peak points for real valued double sequences defined and its relation with statistical convergence of double sequences and statistical monotonicity will be given.

Definition 5.1: (Upper (or Lower) peak point for double sequences) the point x_{jk} is called upper (lower) peak point of the double $x = (x_{jk})$ if $x_{lm} \geq x_{st}$

Theorem 5.1: If the index set of peak points of the double sequence $x = (x_{jk})$ has asymptotic density 1, then the sequence is statistical monotone.

Proof: Let us denote the index set of upper peak points of the double sequence $x = (x_{jk})$ by

$$H = \{j_i k_i : x_{j_i k_i} upper\ peak\ point\ of\ (x_{jk})\} \subset \mathbb{N}.$$

Since $\delta_2(H) = 1$, and $x = (x_{jk})$ is monotone on H, the double sequence $x = (x_{jk})$ is statistical monotone.

Remark 5.1: The inverse of theorem 5.1 is not necessarily true.

Consider $x = (x_{jk})$ where

$$x_{jk} = \begin{cases} \dfrac{1}{mn}, & j,k = m^2n^2\ m,n \in \mathbb{N}, \\ jk, & jk \neq m^2n^2 \end{cases}$$

i.e. $x = (x_{jk}) = \left(1,2,3,\frac{1}{2},5,6,7,8,\frac{1}{3},\ldots\right)$.

Since the set $H = \{m^2, n^2 : m, n \in \mathbb{N}\}$ possesses an asymptotic density $\delta(H) = 0,$ [see Mursaleen (2003)] the sequence $x = (x_{jk})$ is statistical monotone increasing. But, it has no any peak points.

Corollary 5.1: If $x = (x_{jk})$ is bounded and the index set of upper (lower) peak points

$$H = \{j_i k_i : x_{j_1 k_i} upper\ (lower) peak\ point\ of\ (x_{jk})\}$$

Possesses an asymptotic density 1, then $x = (x_{jk})$ is statistical convergent.

Remark 5.2: In Corollary 5.1, ordinary convergence cannot replace statistical convergence.

Consider the sequence $x = (x_{jk})$ where

$$x_{jk} = \begin{cases} -1, & j,k = m^2, n^2, \quad m,n \in \mathbb{N} \\ \dfrac{1}{st} & j,k \neq m^2, n^2 \end{cases}$$

The index set of upper peak points of the square of the sequence $x = (x_j k)$

$\{st : s, t \neq m^2, n^2, m, n \in \mathbb{N}\}$. It is clear that $\delta(H) = 1$ and $x = (x_{jk})$ is bounded.

So, the hypothesis of Corollary 5.1 is fulfilled. Then subsequence

$$(x_{j_i k_i}) = \left(\frac{1}{2},\frac{1}{3},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8},\frac{1}{10},\ldots\right)$$

Is convergence to zero. Also, $x = (x_n)$ is statistical convergent to zero but it is not convergent to zero.

Remark 5.2: For an ordinary single dimensional sequence, any sequence is a subsequence of itself. However this is not the case in the two-dimensional plane (double sequences) as seen in the following example

Example: The sequence

$$x_{n,k} = \begin{cases} 1, & if\ n = k = 0 \\ 1, & if\ n = 0, k = 1 \\ 1, & if\ n = 1, k = 0 \\ 0, & otherwise \end{cases}$$

Contains only two subsequences namely $[y_{n_k}] = 0$ for each n and k and

$$Z_{n_k} = \begin{cases} 1, & if\ n = k = 0 \\ 0 & otherwise \end{cases}$$

neither subsequence is x_{n_k}.

The following proposition which can be easily verified is also worthy stating.

Proposition 5.1: The double sequence $x = (x_{jk})$ is p −convergent to ℓ if and only if evry subsequence of x is p −convergent

VI. $2A$ −generalization of statistical monotonicity

Statistical monotonicity can be generalized by using $2A$ −density of a subset K of $\mathbb{N}$ for RH −regular non-negative summability matrix $A = \left[a_{jk}^{mn}\right]_{j,k=0}^{\infty}, m, n = 0,1,2, \dots$.

Recall $2A$ −density of a subset $E = (i,j) \subseteq \mathbb{N} \times \mathbb{N}$ of

$$\delta_{2A}(K) = \lim_{p,q\to\infty} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn} \quad exists$$

[see Brono et al., (2013)]

$$\lim_{p,q\to\infty} \sum_{j,k\in E} a_{jk}^{mn} = \lim_{p,q\to\infty} \sum_{j,k\in I} a_{jk}^{mn}\, I_E(j,k) = \lim_{p,q\to\infty} (2A \cdot 1_k)^{(pq)}$$

Exists and is finite.

The sequence $x = (x_{jk})$ is $2A$ −statistically convergent to l, if for every ε>0, the set $K_\varepsilon = \{jk \in \mathbb{N}: |x_{jk} - \ell| \geq \varepsilon\}$ posssesses 2A-density zero [see Brono et al (2013)].

Definition 6.1: A sequence $x = (x_{jk})$ is called 2A-statistical monotone, if there exists a subset H of $\mathbb{N} \times \mathbb{N}$ with $\delta_{2A}(H) = 1$ such that the sequence $x = (x_{jk})$ is monotone on H.

Let $2A = (a_{jk}^{mn})$ and $2B = (b_{jk}^{mn})$ be non-negative regular matrices.

Theorem 6.1: If the condition

$$p - \lim_{p,q\to\infty} sup \sum_{j=0}^{p} \sum_{k=0}^{q} \left|a_{jk}^{mn} - b_{jk}^{mn}\right| = 0 \qquad (3)$$

holds. Then $x = (x_{jk})$ is $2A$ −statistical monotone if and only if $x = (x_{jk})$ is $2B$ −statistical monotone.

Proof: For an arbitrary $H \subset \mathbb{N} \times \mathbb{N}$ the inequality

$$0 \leq |(2A \cdot 1_H)(n) - (2B \cdot 1_H)(n)| = \left|\sum_{jk\in H} a_{jk}^{mn} - \sum_{jk\in H} b_{jk}^{mn}\right| \leq \sum_{j,k\in H} \left|a_{jk}^{mn} - b_{jk}^{mn}\right| \leq \sum_{j,k=1}^{\infty} \left|a_{jk}^{mn} - b_{jk}^{mn}\right|$$

holds. Under the condition (3) $\delta_{2A}(H)$ exists if and only if $\delta_{2B}(H)$ exists, and in the case $\delta_{2A}(H) = \delta_{2B}(H)$. Therefore, $2A$ −statistical monotonicity of $x = (x_{jk})$ implies $2B$ −statistical monotonicity vice versa

Let us consider strictly increasing and non-negative sequence$\{\lambda_{2n}\}_{n\in\mathbb{N}}$ and $E = \{\lambda_{2n}\}_{n=0}^{\infty}$. If $A = (a_{jk}^{mn})$ is an RH−summability matrix, then $2A_{\lambda_2} := \left(a_{\lambda_{(2n),i}}\right)$

Is the submatrix of $A = (a_{jk}^{mn})$. Thus, the A_{λ_2} transformation of a sequence $x = (x_{jk})$ as $\left(A_{\lambda_2} x\right)_n = \sum_{j,k=0,0}^{\infty,\infty} a_{\lambda_{(2n),\lambda_{jk}}} (x_{j,k})$.

Since, A_{λ_2} is a row submatrix of $2A$, it is clear that RH-regular whenever $2A$ is a RH-regular summability matrix . For more on RH-regular summability matrices [see Patterson (1999) and (2000)].

Theorem 6.2 Let $2A$ be a RH-summability matrix and let $E = \{\lambda_{2n}\}$ and $F = \{\rho_{2n}\}$ be an infinite subset of $\mathbb{N} \times \mathbb{N}$, if F/E is finite , then $2A_{\lambda_2}$ −statistical monotonicity implies $2A_{\rho_2}$ −statistical monotonicity.

Proof: Assume that F/E is finite, and$x = (x_{jk})$ is A_{λ_2} −statistical monotone sequence. From the assumption there exists a $n_0 \in \mathbb{N}$ such that

$$\{\rho_2(j,k): j,k \geq j_0, k_0\} \subseteq E.$$

It means that there is a monotone increasing sequence $i(j,k)$ such that $\rho_2(j,k) := \lambda_{2i}(j,k)$. So, the $2A_{\rho_2}$ asymptotic density of the set $Z := \{jk \in \mathbb{N}: \rho_2(j,k) := \lambda_{2i}(j,k)\}$ is

$$\lim_{pq\to\infty} \sum_{j,k=0,0}^{\infty,\infty} a_{\rho_{2(n)k}1, Z(jk)} = \lim_{pq\to\infty} \sum_{j,k=0,0}^{\infty,\infty} a_{\lambda_{2(n)k}1, Z(jk)} = 1.$$

This gives us, $x = (x_{jk})$ is a A_{ρ_2} −statistical monotone sequence.

By the Theorem 6.2 we have the following corollaries:

Corollary 6.1: $2A$ −statistical monotone sequence is $2A_{\lambda_2}$ −statistical monotone.

Corollary 6.2: Under the condition of Theorem 6.2, if $E\Delta F$ is finite, then the sequence $x = (x_{jk}), 2A_{\lambda_2}$ −statistical monotone if and only if A_{ρ_2} −statistical monotone.

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