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On Homogeneous Ternary Quadratic Diophantine Equation $z^2 = 15x^2 + y^2$

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Abstract: *The ternary quadratic homogeneous equations representing homogeneous cone given by $z^2 = 15x^2 + y^2$ is analyzed for its non-zero distinct integer points on it. Five different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns namely Polygonal number, Pyramidal number, Octahedral number, Pronic number, Stella Octangular number and Oblong number are presented. Also knowing an integer solution satisfying the given cone, three triples of integers generated from the given solution are exhibited.*

Keywords: *Ternary homogeneous quadratic, integral solutions.*

I. Introduction

The Ternary quadratic Diophantine equations offer an unlimited field for research because of their variety [1-5].For an extensive review of various problems one may refer [6-20]. This communication concerns with yet another interesting ternary quadratic equation $z^2 = 15x^2 + y^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

NOTATIONS USED

- $T_{m,n}$ -Polygonal number of rank n with size m.
- P_m^n - Pyramidal number of rank n with size m.
- Pr_n - Pronic number of rank n.
- SO_n -Stella Octangular number of rank n.
- Obl_n -Oblong number of rank n.
- OH_n - Octahedral number of rank n.
- Tet_n -Tetrahedral number of rank n.
- PP_n - Pentagonal Pyramidal number of rank n

II. Method Of Analysis

The ternary quadratic equation under consideration is

$$z^2 = 15x^2 + y^2 \tag{1}$$

Different patterns of solutions of (1) are illustrated below.

Pattern-I

Consider (1) as

$$15x^2 + y^2 = z^2 * 1 \tag{2}$$

Assume

$$z = a^2 + 15b^2 \tag{3}$$

Write 1as

$$1 = \frac{\{[(7 + 2n - 2n^2) + i\sqrt{15}(2n-1)][(7 + 2n - 2n^2) - i\sqrt{15}(2n-1)]\}}{(8 - 2n + 2n^2)^2} \tag{4}$$

Substituting (3) and (4) in (2) and employing the method of factorization, define

$$y + i\sqrt{15}x = \frac{\{[(7 + 2n - 2n^2) + i\sqrt{15}(2n-1)](a + i\sqrt{15}b)^2\}}{(8 - 2n + 2n^2)}$$

Equating the real and imaginary parts in the above equation, we get

$$x = \frac{\left[(7+2n-2n^2)2ab + (2n-1)(a^2-15b^2)\right]}{(8-2n+2n^2)}$$

$$y = \frac{\left[(7+2n-2n^2)(a^2-15b^2) - 30ab(2n-1)\right]}{(8-2n+2n^2)}$$

Replacing a by $(8-2n+2n^2)A$,b by $(8-2n+2n^2)B$ in the above equation corresponding integer solutions of (1) are given by

$$x = (8-2n+2n^2)\left[(7+2n-2n^2)2AB + (2n-1)(A^2-15B^2)\right]$$

$$y = (8-2n+2n^2)\left[(7+2n-2n^2)(A^2-15B^2) - 30AB(2n-1)\right]$$

$$z = (8-2n+2n^2)^2\left[A^2+15B^2\right]$$

For simplicity and clear understanding, taking n=1 in the above equations, the corresponding integer solutions of(1) are given by

$$x = 8A^2 - 120B^2 + 112AB$$

$$y = 56A^2 - 840B^2 - 240AB$$

$$z = 64A^2 + 960B^2$$

Properties

1) $y(A, A+1) - 7x(A, A+1) + 1024\ \Pr_A \equiv 0$
2) $x(A, 7A^2-4) + y(A, 7A^2-4) + z(A, 7A^2-4) - t_{258,A} + 384\ CP_A^{14} \equiv 0 \pmod{127}$
3) $8x(A, 2A^2-1) + z(A, 2A^2-1) - 128\ Obl_A - 896\ SO_A \equiv 0 \pmod{128}$
4) $x(A,1) - 44\ \Pr_A - t_{74,A} \equiv 0 \pmod{32}$
5) $x(B(B+1), B) + y(B(B+1), B) - z(B(B+1), B) + t_{3842,B} + 256\ P_A^5 \equiv 0 \pmod{1919}$
6) $y(1, B) + 2160\ t_{3,B} - t_{482,B} \equiv 56 \pmod{1079}$

Pattern-II

It is worth to note that 1 in (2) may also be represented as

$$1 = \frac{\left\{\left[(15-4n^2) + i\sqrt{15}(4n)\right]\left[(15-4n^2) - i\sqrt{15}(4n)\right]\right\}}{(15+4n^2)^2}$$

Following the analysis presented above, the corresponding integer solution to (1) are found to be

$$x = (15+4n^2)\left[(15-4n^2)2AB + 4n(A^2-15B^2)\right]$$

$$y = (15+4n^2)\left[(15-4n^2)(A^2-15B^2) - 120ABn\right]$$

$$z = (15+4n^2)^2\left[A^2+15B^2\right]$$

For the sake of simplicity, taking n=1 in the above equations, the corresponding integer solutions of (1) are given by

$$x = 76A^2 - 1140B^2 + 418AB$$

$$y = 209A^2 - 3135B^2 - 2280AB$$

$$z = 361A^2 + 5415B^2$$

Properties

1) $11x(7B^2-4, B) - 4y(7B^2-4, B) - 41154\ CP_B^{14} \equiv 0$
2) $2x(A, 2A^2+1) + y(A, 2A^2+1) + z(A, 2A^2+1) - 722\ \Pr_A + 4332\ OH_A \equiv 0 \pmod{722}$
3) $19x((B+1)(B+2), B) - 4z((B+1)(B+2), B) - 47652\ P_B^3 - t_{86642,B} \equiv 0 \pmod{43319}$
4) $x(A,1) - 196\ Obl_A - t_{242,A} \equiv 1140 \pmod{103}$

5) $x(A^2+1, A) - x(A^2-1, A) 340 \Pr_A \equiv 0 \pmod{532}$

Pattern-III
The general solution of the given equation is

$$x = x(m,n) = 2mn$$
$$y = y(m,n) = 15m^2 - n^2$$
$$z = z(m,n) = 15m^2 + n^2$$

Properties

1) $x(n(n+1), n) + y(n(n+1), n) - z(n(n+1), n) - 4PP_n - 10\Pr_n + t_{26,n} \equiv 0 \pmod{21}$
2) $x(m,2) + y(m,2) + z(m,2) - 80t_{3,m} + 10Obl_m \equiv 0 \pmod{\ }$
3) $z(m,m) = 16m^2$ Perfect Square
4) $x(2n,n) = 4n^2$ Perfect Square
5) $y(m,3) + z(m,3) - 42\Pr_m + t_{26,m} \equiv 0 \pmod{53}$
6) $x(m,1) + y(m,1) + z(m,1) - 30Obl_m \equiv 0 \pmod{28}$

Pattern- IV
Equation (1) can be written as

$$z^2 - y^2 = 15x^2$$

And we get

$$(z+y)(z-y) = 5x * 3x \qquad (5)$$

CASE 1:
Equation (5) can be written as

$$\frac{z+y}{3x} = \frac{5x}{z-y} = \frac{A}{B} \qquad (6)$$

From equation (6),we get two equations

$$3Ax - Bz - By = 0$$
$$5Bx - Az + Ay = 0$$

We get the integer solutions are

$$x = x(A,B) = -2AB$$
$$y = y(A,B) = -3A^2 - 5B^2$$
$$z = z(A,B) = -3A^2 + 5B^2$$

Properties

1) $y(A,A) + z(A,A) - 44\Pr_A + 50Obl_A \equiv 0 \pmod{6}$
2) $x(A,(A+1)(A+2)) + y(A,(A+1)(A+2)) + z(A,(A+1)(A+2)) + 12Tet_A + t_{14,A} \equiv 0 \pmod{5}$
3) $y(B,B) - z(B,B) - 60\Pr_B + 70Obl_B \equiv 0 \pmod{10}$
4) $y(A,2) - 31\Pr_A + t_{70,A} \equiv 20 \pmod{64}$
5) For all the values of A and B, x+y+z is divisible by 2

CASE 2:
Equation (5) can be written as

$$\frac{z-y}{3x} = \frac{5x}{z+y} = \frac{A}{B} \qquad (7)$$

From equation (7), we have two equations

$3xA + yB - zB = 0$

$5xB - yA - zA = 0$

Solve the above two equations, we get the integer solutions are

$x = x(A,B) = -2AB$

$y = y(A,B) = 3A^2 - 5B^2$

$z = z(A,B) = -3A^2 - 5B^2$

Properties

1) $x(2B^2-1,B) + y(2B^2-1,B) + z(2B^2-1,B) + 2SO_B + 10\,\Pr_B \equiv 0 \pmod{10}$

2) $x(A, A(A+1)) - y(A, A(A+1)) + z(A, A(A+1)) + 4PP_A + 12\,Obl_A + t_{14,A} \equiv 0 \pmod{17}$

3) $x(A,-2A) = 4A^2$ *a perfect Square*

4) Each of the following expressions represents a Nasty number

$y(A,A) - z(A,A)$

$3\{z(A,A)\}$

$6\{x(B,B)\}$

2. Generation of integer solutions

Let (x_0, y_0, z_0) be any given integer solution of (1). Then, each of the following triples of integers satisfies (1):

Triple 1: (x_1, y_1, z_1)

$x_1 = 3^n x_0$

$y_1 = \frac{1}{2}\left((9^n + 1^n)y_0 + (9^n - 1^n)z_0\right)$

$z_1 = \frac{1}{2}\left((9^n - 1^n)y_0 + (9^n + 1^n)z_0\right)$

Triple 2 :(x_2, y_2, z_2)

$x_n = \frac{1}{2}\{[5(3)^n - 3(1)^n]x_0 + [(1)^n - 3^n]z_0\}$

$y_{n=y_0}$

$x_n = \frac{1}{2}\{[5(3)^n - 3(1)^n]x_0 + [(1)^n - 3^n]z_0\}$

Triple 3: (x_3, y_3, z_3)

$x_3 = \frac{1}{16}\left\{(8^n + 15(-8)^n)x_0 + ((-8)^n - 8^n)y_0\right\}$

$y_3 = \frac{1}{16}\left\{(15(-8)^n - 15(8)^n)x_0 + (15(8)^n + (-8)^n)y_0\right\}$

$z_3 = 8^n z_0$

III. Conclusion

In this paper, we have presented four different patterns of infinitely many non-zero distinct integer solutions of the homogeneous cone given by. To conclude, one may search for other patterns of solution and their corresponding properties.

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