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Solution Of Initial Value Problem For A Class Of Hyperbolic Equation With Variable Coefficients

Huanhuan Xiong[1], Zhixiang Wang[1], Xiangqing Zhao[2]
[1]*Department of Mathematics, Zhejiang Ocean University, Zhoushan Zhejiang, China, 316022;*
2Key Laboratory of Oceanographic Big Data Mining & Application of Zhejiang Province, Zhoushan Zhejiang, China, 316022.

Abstract: *This paper study initial value problem for a class of hyperbolic equation with variable coefficients.Similar to finding D' Lambert formulaofstring vibrating equation, we decompose the hyperbolic differential operator first, and then reduce thesolution of hyperbolic equation to the solution of characteristic lines.*
Keywords:*Hyperbolic equation with variable coefficients, Initial value problem, Characteristic line, D' Lambert formula*

I. Introduction

For the partial differential equation with variable coefficients:

$$\Box u \equiv (\partial_t^2 - a^2 x\partial_x - a^2x^2\partial^2_x)u = f(x,t), \qquad (0)$$

its discriminant $\Delta = a^2x^2 > 0\ (\forall x \neq 0)$ which implies that itis hyperbolic for $x \neq 0$; and obviously itdegenerates to a second order ordinary differential equation when $x = 0$.

Observing the fact that hyperbolic equation (0) and the string vibrating equation $\Box u \equiv (\partial_t^2 - a^2\partial^2_x)u = f(x,t)$ have similar operator decomposition: $\Delta = (\partial_t + a\partial_x)(\partial_t - a\partial_x)$ [1] and $\Box = (\partial_t + ax\partial_x)(\partial_t - ax\partial_x)$. This fact hints us thatequation can be also solved by reducing to its characteristic line equations $dx/dt = \pm ax$.

In this paper, we studies the initialvalue problem of the above-mentioned partial differential:

$$\begin{cases} (\partial^2_t - a^2x\partial_x - a^2x^2\partial^2_x)u = f(x,t), & x \neq 0, t > 0, \\ u(x,0) = \varphi(x), & x \neq 0, \\ u_t(x,0) = \psi(x), & x \neq 0, \end{cases} \quad (1)$$

and the main results are as follows:

Theorem 1.1If

$$\begin{cases} \varphi \in C^2(0,\infty),\ \psi \in C^1(0,\infty),\ f \in C^2[(0,\infty)\times(0,\infty)], & x > 0, \\ \varphi \in C^2(-\infty,0),\ \psi \in C^1(-\infty,0),\ f \in C^2[(-\infty,0)\times(0,\infty)], & x < 0. \end{cases}$$

Then, the solution of the initial-value problem (1) can be expressed as1

1Since $x = 0$ is a degenerated line,so the boundary condition for $x = 0$ is not needed[2]. As a matter of fact,(1) consists of two initial value problems:

$$\begin{cases} (\partial^2_t - a^2x\partial_x - a^2x^2\partial^2_x)u = f(x,t), & x > 0,\ t > 0, \\ u(x,0) = \varphi(x), & x > 0, \\ u_t(x,0) = \psi(x), & x > 0 \end{cases} \quad \text{and} \quad \begin{cases} (\partial^2_t - a^2x\partial_x - a^2x^2\partial^2_x)u = f(x,t), & x < 0,\ t > 0, \\ u(x,0) = \varphi(x), & x < 0, \\ u_t(x,0) = \psi(x), & x < 0 \end{cases}$$

The two parts of the formula in Theorem 1.1 are solutions corresponding to these two initial value problems.

$$u(x,t)=$$
$$\begin{cases}\dfrac{\varphi(e^{\ln x-at})+\varphi(e^{\ln x+at})}{2}+\dfrac{1}{2a}\displaystyle\int_{\ln x-at}^{\ln x+at}\psi(e^{\xi})\mathrm{d}\xi+\dfrac{1}{2a}\int_0^t\int_{\ln x-a(t-\tau)}^{\ln x+a(t-\tau)}f(e^{\xi},\tau)\mathrm{d}\xi\mathrm{d}\tau, & x>0,\\ \dfrac{\varphi(-e^{\ln(-x)-at})+\varphi(-e^{\ln(-x)+at})}{2}+\dfrac{1}{2a}\displaystyle\int_{\ln(-x)-at}^{\ln(-x)+at}\psi(-e^{\xi})\mathrm{d}\xi+\dfrac{1}{2a}\int_0^t\int_{\ln(-x)-a(t-\tau)}^{\ln(-x)+a(t-\tau)}f(-e^{\xi},\tau)\mathrm{d}\xi\mathrm{d}\tau, & x<0.\end{cases}$$

II. The preliminary Results

By the superposition principle, (1) can be divided into the following three initial value problems:

$$\begin{cases}\Box \mathrm{u}_1=0,\ x\neq 0, t>0,\\ u_1(x,0)=0,\ x\neq 0,\\ \partial_t u_1(x,0)=\psi(x),\ x\neq 0,\end{cases}\tag{2}$$

$$\begin{cases}\Box \mathrm{u}_2=0,\ x\neq 0, t>0,\\ u_2(x,0)=\phi(x),\ x\neq 0,\\ \partial_t u_2(x,0)=0,\ x\neq 0,\end{cases}\tag{3}$$

and

$$\begin{cases}\Box \mathrm{u}_3=f(x,t),\ x\neq 0, t>0,\\ u_3(x,0)=0,\ x\neq 0,\\ \partial_t u_3(x,0)=0,\ x\neq 0.\end{cases}\tag{4}$$

Lemma 2.1 ∂_t and $\Box$ can be exchanged order, i.e. $\partial_t\Box=\Box\partial_t$.

Indeed, we have

$$\begin{aligned}\partial_t\Box &=\partial_t(\partial_t^2-a^2x\partial_x-a^2x^2\partial_x^2)\\ &=\partial_t\partial_t^2-a^2x\partial_t\partial_x-a^2x^2\partial_t\partial_x^2,\\ \Box\partial_t &=(\partial_t^2-a^2x\partial_x-a^2x^2\partial_x^2)\partial_t\\ &=\partial_t^2\partial_t-a^2x\partial_x\partial_t-a^2x^2\partial_x^2\partial_t\\ &=\partial_t\partial_t^2-a^2x\partial_t\partial_x-a^2x^2\partial_t\partial_x^2.\end{aligned}$$

Proposition 2.1 Assume $u_1=M_{\psi}(x,t)$ is the solution of (2), then the solution of initial value problems (3), (4) can be expressed as:

$$u_2=\frac{\partial}{\partial t}M_{\varphi}(x,t),\tag{5}$$

$$u_3=\int_0^t M_{f_\tau}(x,t-\tau)\mathrm{d}\tau.\tag{6}$$

where $f_\tau=f(x,\tau)$ and $M_{\varphi}(x,t)$, $M_{f_\tau}(x,t-\tau)$ are sufficiently smooth with respect to x, t and τ respectively.

Proof: Weshow (5) satisfies (3) first. According to the assumption, it is known M_{φ} satisfies

$$\begin{cases} \square \mathrm{M}_{\phi} = 0, & (7) \\ \mathrm{M}_{\phi}(x,0) = 0, & (8) \\ \dfrac{\partial}{\partial t}\mathrm{M}_{\phi}(x,0) = \phi(x). & (9) \end{cases}$$

Thus, by Lemma 2.1, andby (7) - (9), we have

$$\square \mathrm{u}_2 = \square \frac{\partial}{\partial \mathrm{t}} M_{\phi} = \frac{\partial}{\partial \mathrm{t}} \square M_{\phi} = 0,$$

$$\frac{\partial}{\partial t} u_2(x,0) = \frac{\partial^2}{\partial t^2} M_{\varphi} |_{t=0} = (a^2 x \partial_x + a^2 x^2 \partial_{xx}) M_{\varphi} |_{t=0} = 0,$$

$$u_2(x,0) = \frac{\partial}{\partial t} M_{\varphi}(x,0) = \varphi(x).$$

Then we prove (6) satisfies (4). Notice that $w = M_{f_\tau}(x, t-\tau)$ satisfies

$$\begin{cases} \square \mathrm{w} = 0, & (10) \\ \mathrm{w}\big|_{t=\tau} = 0, & (11) \\ \dfrac{\partial}{\partial t}\mathrm{w}\big|_{t=\tau} = f(x,\tau), & (12) \end{cases}$$

Thus by (10) - (12), we have

$$u_3(x,0) = 0,$$

$$\frac{\partial}{\partial t} u_3 = M_{f_\tau}(x, t-\tau) |_{\tau=t} + \int_0^t \frac{\partial}{\partial t} M_{f_\tau}(x, t-\tau) \mathrm{d}\tau$$

$$= \int_0^t \frac{\partial}{\partial t} M_{f_\tau}(x, t-\tau) \mathrm{d}\tau.$$

Then

$$\frac{\partial}{\partial t} u_3(x,0) = 0.$$

As well as

$$\frac{\partial^2 u_3}{\partial t^2} = \frac{\partial}{\partial t} M_{f\tau}(x, t-\tau) |_{\tau=t} + \int_0^t \frac{\partial^2}{\partial t^2} M_{f\tau}(x, t-\tau) \mathrm{d}\tau$$

$$\overset{(13)}{=} f(x,t) + a^2 \int_0^t (x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x}) M_{f\tau}(x, t-\tau) \mathrm{d}\tau$$

$$= f(x,t) + a^2 (x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x}) \int_0^t M_{f\tau}(x, t-\tau) \mathrm{d}\tau.$$

And so

$$\frac{\partial^2}{\partial t^2} u_3 = f(x,t) + a^2 (x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x}) u_3.$$

III. Solution of Characteristic Lines

According to Proposition 2.1, it is enough to solve initial value problems (2) in detail.To apply the characteristic line theory to solve (2), we give the operator decomposition first:

Lemma 3.1. The operator $\square$ can be decomposed into

$\square = (\partial_{\mathrm{t}} + ax\partial_{\mathrm{x}})(\partial_{\mathrm{t}} - ax\partial_{\mathrm{x}}).$ (13)

Proof: Indeed

$$
\begin{aligned}
&(\partial_{\mathrm{t}} + ax\partial_{\mathrm{x}})(\partial_{\mathrm{t}} - ax\partial_{\mathrm{x}}) \\
&= \partial_{\mathrm{t}}(\partial_{\mathrm{t}} - ax\partial_{\mathrm{x}}) + ax\partial_{\mathrm{x}}(\partial_{\mathrm{t}} - ax\partial_{\mathrm{x}}) \\
&= \partial_t^{\,2} - ax\partial_{\mathrm{t}}\partial_{\mathrm{x}} + ax\partial_{\mathrm{x}}\partial_{\mathrm{t}} - a^2 x\partial_{\mathrm{x}} - a^2x^2\partial_x^{\,2} \\
&= \partial_t^{\,2} - a^2 x\partial_{\mathrm{x}} - a^2x^2\partial_x^{\,2} \\
&= \Box.
\end{aligned}
$$

Proposition 3.1 Initial Value Problem (2) has a solution

$$
u_1(x,t) = \begin{cases} \dfrac{1}{2a}\displaystyle\int_{\ln x - at}^{\ln x + at} \psi(e^{\xi})\mathrm{d}\xi, \; x > 0, \\ \dfrac{1}{2a}\displaystyle\int_{\ln(-x) - at}^{\ln(-x) + at} \psi(-e^{\xi})\mathrm{d}\xi, \; x > 0. \end{cases} \qquad (14)
$$

Proof: By Lemma 3.1,

$$
\begin{cases} \Box \mathrm{u} = 0, \\ u(x,0) = 0, \\ u_t(x,0) = \psi(x). \end{cases} \qquad (15)
$$

can be divided into the following two initial value problem of first order equations:

$$
\begin{cases} \partial_t u - ax\partial_x u = v, \\ u(x,0) = 0, \end{cases} \qquad (16)
$$

$$
\begin{cases} \partial_t v + ax\partial_x v = 0, \\ v(x,0) = \partial_t u(x,0) - ax\partial_x u(x,0) = \psi(x). \end{cases} \qquad (17)
$$

Let

$$
\frac{dx}{dt} = -ax, \; \frac{dx}{dt} = ax.
$$

Then the characteristic line of (16), (17) are solved as:

$$
x = x_1(t) = \begin{cases} e^{c-at}, & x > 0, \\ -e^{c-at}, & x < 0, \end{cases} \qquad (18)
$$

$$
x = x_2(t) = \begin{cases} e^{c+at}, & x > 0, \\ -e^{c+at}, & x < 0. \end{cases} \qquad (19)
$$

Along the characteristic line $x_1(t)$, (16) can be transformed into

$$
\frac{\mathrm{d}u[x_1(t),t]}{\mathrm{d}t} = v[x_1(t),t]. \qquad (20)
$$

Along the characteristic line $x_2(t)$, (17) can be transformed into

$$
\frac{\mathrm{d}v[x_2(t),t]}{\mathrm{d}t} = 0. \qquad (21)
$$

By (17), (21) , we obtain:

$$
v[x_2(t),t] = v[x_2(0),0] = \psi(x_2(0)). \qquad (22)
$$

Since $x_2(t) = x$, and

$$
\begin{aligned}
x_2(0) &= \begin{cases} e^{c}, & x > 0, \\ -e^{c}, & x < 0, \end{cases} \\
&= \begin{cases} e^{\ln x - at}, & x > 0, \\ -e^{\ln(-x) - at}, & x < 0, \end{cases}
\end{aligned}
$$

the solution of (17) is indeed as follows:

$$v(x,t)=\begin{cases}\psi(e^{\ln x-at}), & x>0,\\ \psi(-e^{\ln(-x)-at}), & x>0.\end{cases}$$

Substitute it into (20) , then by (16) , we obtain

$$u[x_1(t),t]=\begin{cases}\int_0^t\psi(e^{\ln x_1(\tau)-a\tau})\mathrm{d}\tau, & x>0,\\ \int_0^t\psi(-e^{\ln(-x_1(\tau))-a\tau})\mathrm{d}\tau, & x<0,\end{cases}$$
$$=\begin{cases}\int_0^t\psi(e^{c-2a\tau})\mathrm{d}\tau, & x>0,\\ \int_0^t\psi(-e^{c-2a\tau})\mathrm{d}\tau, & x<0.\end{cases}$$

By (18), we obtain the solution of (17) as follows:
When $x>0$,

$$\begin{aligned}u_1(x,t)&=\int_0^t\psi(e^{c-2a\tau})\mathrm{d}\tau\\&=-\frac{1}{2a}\int_c^{c-2at}\psi(e^{\xi})\mathrm{d}\xi \qquad (23)\\&=\frac{1}{2a}\int_{\ln x-at}^{\ln x+at}\psi(e^{\xi})\mathrm{d}\xi.\end{aligned}$$

When $x>0$,

$$\begin{aligned}u_1(x,t)&=\int_0^t\psi(-e^{c-2a\tau})\mathrm{d}\tau\\&=-\frac{1}{2a}\int_c^{c-2at}\psi(-e^{\xi})\mathrm{d}\xi \qquad (24)\\&=\frac{1}{2a}\int_{\ln(-x)-at}^{\ln(-x)+at}\psi(-e^{\xi})\mathrm{d}\xi.\end{aligned}$$

IV. The Proof of Theorem 1.1

By Proposition 2.1, 3.1, we obtain:
When $x>0$,

$$\begin{aligned}u_2(x,t)&=\frac{\partial}{\partial t}[\frac{1}{2a}\int_{\ln x-at}^{\ln x+at}\varphi(e^{\xi})\mathrm{d}\xi]\\&=\frac{1}{2a}[a\varphi(e^{\ln x+at})-(-a)\varphi(e^{\ln x-at})] \qquad (25)\\&=\frac{\varphi(e^{\ln x+at})+\varphi(e^{\ln x-at})}{2},\end{aligned}$$

$$\begin{aligned}u_3(x,t)&=\int_0^t\frac{1}{2a}\int_{1n(-x)-a(t-\tau)}^{1n(-x)+a(t-\tau)}f(-e^{\xi},\tau)d\xi d\tau \qquad (26)\\&=\frac{1}{2a}\int_0^t\int_{1n(-x)-a(t-\tau)}^{1n(-x)+a(t-\tau)}f(-e^{\xi},\tau)d\xi d\tau.\end{aligned}$$

When $x<0$,

$$\begin{aligned}u_2(x,t)&=\frac{\partial}{\partial t}\left[\frac{1}{2a}\int_{1n(-x)-at}^{1n(-x)+at}\varphi(-e^{\xi})d\xi\right]\\&=\frac{1}{2a}[a\varphi(-e^{1n(-x)+at})-(-a)\varphi(-e^{1n(-x)-at})] \qquad (27)\\&=\frac{\varphi(-e^{1n(-x)+at})+\varphi(-e^{1n(-x)-at})}{2}.\end{aligned}$$

$$u_3(x,t)=\int_0^t \frac{1}{2a}\int_{\ln(-x)-a(t-\tau)}^{\ln(-x)+a(t-\tau)} f(-e^{\xi},\tau)d\xi d\tau \qquad (28)$$
$$=\frac{1}{2a}\int_0^t \int_{\ln(-x)-a(t-\tau)}^{\ln(-x)+a(t-\tau)} f(-e^{\xi},\tau)d\xi d\tau.$$

By the superposition principle, the solution of (1) is

$$u(x,t)=u_1(x,t)+u_2(x,t)+u_3(x,t)$$
$$=\begin{cases} \dfrac{\varphi(e^{\ln x-at})+\varphi(e^{\ln x+at})}{2}+\dfrac{1}{2a}\displaystyle\int_{\ln x-at}^{\ln x+at}\psi(e^{\xi})\mathrm{d}\xi+\dfrac{1}{2a}\int_0^t\int_{\ln x-a(t-\tau)}^{\ln x+a(t-\tau)} f(e^{\xi},\tau)\mathrm{d}\xi\mathrm{d}\tau, & x>0, \\ \dfrac{\varphi(-e^{\ln(-x)-at})+\varphi(-e^{\ln(-x)+at})}{2}+\dfrac{1}{2a}\displaystyle\int_{\ln(-x)-at}^{\ln(-x)+at}\psi(-e^{\xi})\mathrm{d}\xi+\dfrac{1}{2a}\int_0^t\int_{\ln(-x)-a(t-\tau)}^{\ln(-x)+a(t-\tau)} f(-e^{\xi},\tau)\mathrm{d}\xi\mathrm{d}\tau, & x<0. \end{cases}$$

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Corresponding author: HuanhuanXiong

Address: No.1, Haida South Road, Lincheng Changzhi Island,Zhoushan, Zhejiang, 316022,China

E-mail：xionghuanhuan1995@163.com