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On Progenerator Semimodules

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***Abstract**: The Generator and Progenerator plays a vital role in the study of category equivalences. In this paper we establish the crutial result that an progrnerator semimodules are preserved undercategory equivalences.*
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I. Introduction

Recall that a set R together with two binary operations addition (+) and multiplication (.) is called a semiring, provided $(R,+)$ is an additive abelian monoid with identity element 0_R, $(R,\cdot)$ is a semi group and multiplication distributes over addition from left and from the right.

As usual a left R-semimodule M is a commutative monoid $(M, +)$ with additive identity o_M for which we have a function $R \times M \to M$ defined by $(r,m) \to r.m$ and called scalar multiplication which satisfies the following conditions for all r,r' of R and all elements m, m' of M,

1. $(r + r')m = rm + r'm$
2. $r(m + m') = rm + rm'$
3. $(r \cdot r')m = r(r' \cdot m)$
4. $1_R \cdot m = m$
5. $0_R \cdot m = 0_M = r.0_M$

By a semiring R we always mean a semiring R with 1. Homomorphisms are acting on the left and not as action on the right as in [2]. $smod-R$, $S-smod$, $csmod-R$ and $S-csmod$ respectively denote the categories of right R-semimodules, left S-semimodules, additively cancellative right R-semimodules and additively cancellative left S-semimodules.

A left $R-$ semimodule M *is* a generator of the category R-smod if for every semimodule N there exists a surjective morphism $M^{(I)} \to N$ for a suitable set I. For any R- semimodule M, consider the subset $I_R(M)$ of R consisting of the elements of the form $\sum_{i=1}^{n} f_i(m_i)$ where f_i from $Hom(M,R)$ and the m_i are from M. $I_R(M)$ is a two sided ideal of R, called the **trace ideal** of M.

Lemma 1.1[3] : *M is an R-generator if and only if the functor $Hom(M, -)$ is faithful.*
Theorem 1.2[3] : *Let M be a left R-semimodule, Then the following conditions are equivalent.*
a) There exists $f_1, f_2, \ldots \ldots f_n \in Hom_R(M,R)$ *and* $m_1, m_2, \ldots, m_n \in M$ *with*, $\sum_{i=1}^{n} f_i(m_i) = 1$
b) The Trace ideal $I_R(M) = R$.
c) The functor $Hom(M,-)$ is faithful.

Proposition 1.*3[3] : The generator semimodules are preserved under any category equivalence, i.e., if F is a category equivalence from one category of semimodules say C to another category of semimodules say D, them a semimodule M is a generator in the first category if and only if F (M) is a generator in the second category.*

II. Progenerator Semimodules

*A R-semimodule M is an R-**progenerator** if M is a finitely generated, projective and generator over R.*
Lemma 2.1 : *Let C and D be categories of semimodules under the inverse equivalence $F : C \to D$ and $G : D \to C$. Then for any objects L,L' in C, the homomorphism μ: $Hom_C(L\ L') \to Hom_D(F(L), F(L'))$ given by $\mu(g) = F(g)$ is one-one and onto.*

In any category, a map $f: M \to N$ is called a proper monomorphism if f is a monomorphism but there is no map $g : N \to M$ such that $fg = Id_N$ and $gf = Id_M$. Two monomorphisms $f : M \to N$ and $g: L \to N$ are ordered $f \leq g$ if and only if there exists a map $h: M \to L$ such that $f = gh$.

The concept of a sub object of an object in a category provides a categorical way of dealing with the subsemimodules of a given semimodule N. In C, a subobject of the semimodule N is an equivalence class $[f]$ of monics $f : M \rightarrow N$ where the equivalence relation is defined by $f \sim f' : M' \rightarrow N$ if there exists an isomorphism $g : M' \rightarrow M$ such that $f' = fg$. In this case $f'(M') = f(M)$, so all of the f in $[f]$ have the same image in N and this is a subsemimodule. Moreover, if M is any subsemimodule, then we have the injection $i: M \rightarrow N$, and $i(M) = M$. Thus we have a bijection $\theta : A \rightarrow B$ where A is the set of all subobjects of N and B is the set of all subsemimodules of N, under the definition $\theta([f]) = f(M)$.

This is order preserving if subsemimodules are ordered in the usual way by inclusion, and we define $[f'] \leq [f]$ for subobjects of N to mean $f' = fg$ for a monomorphism g. If $\{N_\alpha\}$ is a directed set of subsemimodules of N, then $\bigcup N_\alpha$ is a subsemimodule and this is a sup for the set of all N_α in the partial ordering by inclusion. It follows that any directed set of subobjects of N has a sup in the ordering of subobjects.

If (F, G) is an equivalence of C and D and $\{[f_\alpha]\}$ is a directed set of subobjects of N with $Sup[f]$, then it is clear that $\{[F(f_\alpha)]\}$ is a directed set of subobjects of $F(N)$ with $Sup[F(f)]$. A subobject $[f]$ is called proper if f is not an isomorphism or equivalently if $f(M)$ for $f: M \rightarrow N$ is a proper subsemimodule of N. If $[f]$ is proper then $[F(f)]$ is proper.

Proposition 2.2: *In any category consisting of all the semimodules over some semiring, the following statements about a fixed semimodule M are equivalent.*

a) *M is finitely generated.*
b) *The union of every linearly ordered chain of proper subsemimodule of M is a proper subsemimodule.*
c) *Every linearly ordered chain of proper monomorphism into M possesses an upper bound in the collection of all proper monomorphism into M.*

Proposition 2.3: *The finitely generated semimodules are preserved under any category equivalence, i.e., if F is a category equivalence from one category of semimodules say C to another category of semimodules say D , then a semimodule M is finitely generated in the first category if and only if F (M) is finitely generated in the second category.*

Proof: *Proof follows by above discussion and Proposition 1.2.*
Thus finitely generated semimodules are preserved under any category equivalence. Therefore F preserves the subcategory of free semimodules with finite basis set whence , as projective semimodules are retracts of free semimodules (discussed in [3]), we have the following.

Proposition 2.4: *The projective semimodules are preserved under any category equivalence, i.e., if F is a category equivalence from one category of semimodules say C to another category of semimodules say D , then a semimodule M is projective in the first category if and only if F (M) is projective in the second category.*

Theorem 2.5: *The progenerator semimodules are preserved under any category equivalence,If smod−R and S−smod are equivalent under a category equivalence F, then a semimodule M in smod−R is an R-progenerator if and only if F(M) is an S-progenerator.*

References

[1]. H. M. J. Al-Thani : Flat Semimodules, UMMS, 17, 2004, 873-880.
[2]. F. DeMayer and E. Ingrahm : Separable Algebras over Commutative Rings, V-181, Springer - Verlag, New York, 1971.
[3]. Pritam R. Gujarathi: A Note on Generator Semimodules, International Journal of Mathematical Sciences and its Applications, 1(3), 2011
[4]. P. R. Gujarathi and R.P. Deore A Note on Localization of Semimodules, Inter- national Journal of Algenra, 15(4) 2010, 721-725
[5]. R. P. Deore and K. B. Patil : On The Dual Basis for Projective Semimodules and its Applications, Sarajevo J. Maths , 1(14) 2005, 161-169.
[6]. J. S. Golan: The Theory of Semirings with applications in Mathematics and The- oretical Computer Sciences, Pitman Monographs and surveys in Pure and Applied Mathematics 54, Longman Science Tech. Harlow-1992.
[7]. U. Hebisch and W.J.Weinert: Semirings, World Scientific, New Jersey 1998. [8] N. Jacobson : Basic Algebra-II, HPC. Ltd. 1983.
[8]. V. S. Krishnan : An Introduction to Category Theory, Elsevier North Holland , 1981.
[9]. B. Pareigis : Categories and Functors , Academic Press New York 1970.
[10]. K.B. Patil and R. P. Deore: Some Results on Semirings and Semimodules, Bull.
[11]. Cal. Math.Soc. 98, (1) 49-56, 2006.
[12]. Olga Sokratova: On Semimodules over commutative, Additively Idempotent Semirings, Semigroups Foram 2000.