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Patterns of Continued Fractions with a Positive Integer as a Gap

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Abstract : *In this paper we indentify the patterns of continued fractions of rational numbers* p/q *with* $k = p - q$. *Here we try to find minimum number of distinct patterns that will give rise to the remaining patterns.*
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Notations:

1. $\langle a_0, a_1, a_2, a_3, \cdots a_n \rangle$ *Continued fraction expansion.*
2. $\left[\dfrac{k}{m_1}\right]$ *Integer part of the rational number* k/m_1.
3. $\varphi(k)$ *Euler's totient function.*
4. $\gcd(k, m_1)$ *Greatest common devisor of* k *and* m_1.

I. Introduction

The origin of continued fractions is traditionally placed at the time of the creation of Euclid's Algorithm. Due to its close relationship to continued fraction, the creation of Euclid's Algorithm signifies the initial development of continued fractions. Euler showed that every rational can be expressed as a terminating simple continued fraction. He also provided an expression for e in continued fraction form. He used this expression to show that e and e^2 are irrational. Any finite simple continued fraction represents a rational number. Conversely any rational number can be expressed as a finite simple continued fraction, and in exactly two ways.

First we give different representations of a rational number as a continued fraction.

An expression of the form

$$\frac{p}{q} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots}}}}$$

where a_i, b_i are real or complex numbers is called a continued fraction.

An expression of the form

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where $b_i = 1\ \forall i,$ and $a_0, a_1, a_2, \cdots$ are each positive integers is called a simple continued fraction.

The continued fraction is commonly expressed as

$$\frac{p}{q}=a_0+\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots \text{ or simply as } \langle a_0, a_1, a_2, a_3, \cdots\rangle .$$

The elements $a_0, a_1, a_2, a_3, \cdots$ are called the partial quotients. If there are finite number of partial quotients, we call it finite simple continued fraction, otherwise it is infinite. We have to use either Euclidean algorithm or continued fraction algorithm to find such partial quotients.

One essential tool in studying the theory of continued fraction is the study of convergent of a continued fraction. Some simple concepts used in this paper are given below.

If $a_0, a_1, a_2, a_3, \cdots$ is an infinite sequence of positive integers, except a_0 (a_0 may or may not be zero), we define two sequence of integers $\{h_n\}$ and $\{k_n\}$ inductively as follows.

$$h_{-2}=0, h_{-1}=1 \qquad\qquad k_{-2}=1, k_{-1}=0$$

$$h_i=a_i h_{i-1}+h_{i-2}\ \forall\, i\geq 0 \qquad\qquad k_i=a_i k_{i-1}+k_{i-2}\ \forall\, i\geq 0$$

Then as proved in [5] for any positive real number x, $\langle a_0, a_1, a_2, \ldots, a_{n-1}, x\rangle=\dfrac{xh_{n-1}+h_{n-2}}{xk_{n-1}+k_{n-2}}$. (1)

Also it is noted in [5] that if we define $r_n=\langle a_0, a_1, a_2, \ldots, a_n\rangle$ for all integers $n\geq 0$, then $r_n=\dfrac{h_n}{k_n}$. (2)

It is observed in [6] that the rational number

$$\langle a_0, a_1, a_2, ..a_n\rangle=\frac{h_n}{k_n}=r_n \tag{3}$$

is called the n^{th} convergent to the infinite continued fraction.

II. Method Of Analysis

In this paper we try to find some patterns of continued fractions of rational number p/q with gap k, where k is any positive integer which represents the difference between p and q.

Any rational number p/q may be considered in the form

$$\frac{kn+m_1}{kn+k+m_1}, where\ 1\leq m_1\leq k-1, and\ \gcd(k, m_1)=1,\ n=1,2,3,....$$

We discuss the patterns based on the value of m_1. Four possible values for m_1 are identified. They are given below.

If $m_1>k$, m_1 can be take any one of the values $k+1, k+2, k+3, ..2k-1$ and hence the numerators of the rational numbers are congruent to $1,2,3,...k-1$ modulo k, which falls in any one of the four cases.

Case (i): When $m_1=1$, the continued fraction of

$$\frac{kn+1}{kn+k+1} \quad is \quad \langle 0,1,n,k\rangle$$

Case (ii): When $1<m_1<k-1$, the continued fraction of

$$\frac{kn+m_1}{kn+k+m_1}\ is\ \left\langle\ 0,1,n,\left[\frac{k}{m_1}\right],\left[\frac{m_1}{m_2}\right],.....,\left[\frac{m_{r-1}}{m_r}\right]\ \right\rangle,\ \ m_r=1.$$

Case (iii): When $m_1=k-1$, the continued fraction of

$$\frac{kn+(k-1)}{kn+2k-1}\ is\ \langle 0,1,n,1,\ k-1\rangle$$

Case (iv): When $m_1=k-m_1$, and $1<m_1<k-1$, the continued fraction of

$$\frac{kn+k-m_1}{kn+2k-m_1} \ is \ \left\langle 0,1,n,1,\left[\frac{k}{m_1}\right]-1,\left[\frac{m_1}{m_2}\right],.....,\left[\frac{m_{r-1}}{m_r}\right] \right\rangle \ , \ \ m_r = 1.$$

Properties observed of the rational numbers discussed in the above four cases are:

(a) The numerator of rational number expressed in case (i) is congruent to 1 modulo k and in case (ii) is congruent to m_1 modulo k whereas in case (iii) and case (iv) the numerators are congruent to $(k-1)$ and $(k-m_1)$ modulo k respectively.

(b) In case (i) and case (iii) the first three partial quotients are same. The fourth partial quotient k of case (i) is subdivided into 1 and $(k-1)$ in case (iii). Similarly the fourth partial quotient $\left[\frac{k}{m_1}\right]$ of case (ii) is subdivided into 1 and $\left[\frac{k}{m_1}\right]-1$ in case (iv) (the remaining partial quotients are fixed).

(c) The sum of the numerators of case (i) and case (iii) is congruent to zero modulo k. Similar results hold for case (ii) and case (iv).

(d) Since $\gcd(k,m_1)=1$ and k depends on choice of m_1, the number of patterns exist is $\varphi(k)$.

Since $\varphi(k)$ is even, we can pair the patterns in such a way that the sum of the numerators of such pairs are divisible by k and we say that the pairs are related.

2.1 THEOREM:

If the continued fraction expansion $\langle a_0, a_1, a_2, ...a_n \rangle$ represents the rational number $\frac{p}{q}$ then the continued fraction expansion $\langle 1, a_0-1, a_1, a_2, ...a_n \rangle$ represents the rational number $\frac{p}{p-q}$.

Proof:

Consider

$$\langle 1, a_0-1, a_1, a_2, ...a_n \rangle = 1+\cfrac{1}{-1+\left[a_0 \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}}\right]} = 1+\cfrac{1}{-1+\cfrac{p}{q}} = 1+\frac{q}{p-q} = \frac{p}{p-q}.$$

2.2 THEOREM:

For any positive integer k, if the continued fraction expansion $\langle 0,1,n,k \rangle$ represents the rational number $\frac{kn+1}{kn+(k+1)}$, where n is a positive integer then the continued fraction expansion $\langle 0,1,n,\,1,k-1 \rangle$ represents the rational number $\frac{kn+(k-1)}{kn+2k-1}$, where n is a positive integer.

Proof:

Let $\frac{kn+1}{kn+(k+1)} = \frac{p}{q}$ and $\frac{kn+(k-1)}{kn+2k-1} = \frac{p'}{q'}$.

Since the continued fractions expansion are unique,

$$\langle 0,1,n,k \rangle = \left\langle 0,1,n+\frac{1}{k} \right\rangle$$

$$= \left\langle 0,1,\frac{kn+1}{k}\right\rangle$$

$$= \left\langle 0,\ 1+\frac{k}{kn+1}\right\rangle$$

$$= \left\langle 0,\ \frac{kn+k+1}{kn+1}\right\rangle$$

$$= \left\langle 0+\frac{kn+1}{kn+k+1}\right\rangle$$

$$= \frac{kn+1}{kn+(k+1)} = \frac{p}{q}$$

Hence the continued fraction expansion $\langle 0,1,n,k\rangle$ represents the rational number $\frac{kn+1}{kn+(k+1)}$.

Suppose $\langle 0,1,n,\ 1,k-1\rangle = \langle 0,1,n,\ x\rangle$, where $x = \langle 1,\ k-1\rangle = 1+\frac{1}{k-1} = \frac{k}{k-1}$.

Using (1), $\langle 0,1,n,\ 1,k-1\rangle = \langle 0,1,n,\ x\rangle = \frac{xp_{n-1}+p_{n-2}}{xq_{n-1}+q_{n-2}}$.

Since the convergent of $\frac{p}{q}$ and $\frac{p'}{q'}$ are same upto $n \le 2,$ where $\frac{p_n}{q_n}$ is the n^{th} convergent of $\frac{p}{q}$ and $\frac{p'_n}{q'_n}$ is the n^{th} convergent of $\frac{p'}{q'}$ we get,

$$\langle 0,1,n,\ x\rangle = \frac{xp_2+p_1}{xq_2+q_1}.$$

Since $p_0 = q_0 = 0$, $p_1 = q_1 = 1$, $p_2 = n\, and\, q_2 = n+1.$

$$\langle 0,1,n,\ 1,k-1\rangle = \frac{\frac{k}{k-1}n+1}{\frac{k}{k-1}(n+1)+1}$$

$$= \frac{kn+(k-1)}{kn+2k-1} = \frac{p'}{q'}$$

Hence the continued fraction expansion $\langle 0,1,n,\ 1,k-1\rangle$ represents the rational number $\frac{kn+(k-1)}{kn+2k-1}$, where n is a positive integer.

2.3 THEOREM:

For any positive integer $k,$ if the continued fraction expansion $\left\langle 0,1,n,\left[\frac{k}{m_1}\right],\left[\frac{m_1}{m_2}\right],\ldots..,\left[\frac{m_{r-1}}{m_r}\right]\right\rangle$, $m_r = 1$ represents the rational number $\frac{kn+m_1}{kn+(k+m_1)}$, where n is a positive integer then the continued fraction expansion

$\left\langle 0,1,n,1,\left[\frac{k}{m_1}\right]-1,\left[\frac{m_1}{m_2}\right],.....,\left[\frac{m_{r-1}}{m_r}\right]\right\rangle$, $m_r = 1$ represents the rational number $\frac{kn+(k-m_1)}{kn+2k-m_1}$, *where* $1 < m_1 < k$ *and* $\gcd(k,m_1)=1$, n is a positive integer.

Proof:

$$Let\ \frac{p}{q}=\langle 0,1,n,x\rangle, \quad where\ x=\left\langle \left[\frac{k}{m_1}\right],\left[\frac{m_1}{m_2}\right],.....,\left[\frac{m_{r-1}}{m_r}\right]\right\rangle.$$

$$Let\ \frac{p'}{q'}=\langle 0,1,n,y\rangle, \quad where\ y=\left\langle 1,\left[\frac{k}{m_1}\right]-1,\left[\frac{m_1}{m_2}\right],.....,\left[\frac{m_{r-1}}{m_r}\right]\right\rangle.$$

Comparing the continued fractions of x and y, and using (theorem 2.1) we get

$$y=\frac{k}{k-m_1}.$$

Hence $$\langle 0,1,n,y\rangle=\langle 0,1,\ n,\frac{k}{k-m_1}\rangle$$

Proceeding as the previous theorem we get

$$\langle 0,1,n,y\rangle=\frac{kn+k-m_1}{kn+2k-m_1}.$$

Hence the continued fraction expansion $\left\langle 0,1,n,1,\left[\frac{k}{m_1}\right]-1,\left[\frac{m_1}{m_2}\right],.....,\left[\frac{m_{r-1}}{m_r}\right]\right\rangle$, $m_r = 1$ represents the rational number $\frac{kn+(k-m_1)}{kn+2k-m_1}$, where n is a positive integer.

III. Illustration

The following table gives the patterns of the continued fraction of the rational numbers with $k = 11$. Here the number of patterns exist is $\varphi(k)=10$. Let it be $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}$.

Rational number	**Continued fraction expansion**
$\frac{11n+1}{11n+12}$	$p_1 : \langle 0,1,n,11\rangle$
$\frac{11n+2}{11n+13}$	$p_2 : \langle 0,1,n,5,2\rangle$
$\frac{11n+3}{11n+14}$	$p_3 : \langle 0,1,n,3,1,2\rangle$
$\frac{11n+4}{11n+15}$	$p_4 : \langle 0,1,n,2,1,3\rangle$
$\frac{11n+5}{11n+16}$	$p_5 : \langle 0,1,n,2,5\rangle$
$\frac{11n+6}{11n+17}$	$p_6 : \langle 0,1,n,1,1,5\rangle$
$\frac{11n+7}{11n+18}$	$p_7 : \langle 0,1,n,1,1,1,3\rangle$
$\frac{11n+8}{11n+19}$	$p_8 : \langle 0,1,n,1,2,1,2\rangle$

$\frac{11n+9}{11n+20}$	$p_9 : \langle 0,1,n,1,4,\ 2\rangle$
$\frac{11n+10}{11n+21}$	$p_{10} : \langle 0,1,n,1,10\rangle$

Here the patterns $p_1, p_{10};\ p_2, p_9;\ p_3, p_8;\ p_4, p_7,;\ p_5, p_6$ are related. Hence it is enough to find the patterns p_1 to p_5 while the remaining patterns p_6 to p_{10} are found by the procedure stated in theorem (2.2 and 2.3).

IV. Conclusion

The distinct patterns of continued fractions of rational number p/q with $k = p - q$ are found here. To find all $\varphi(k)$ patterns of continued fractions of the rational number p/q with gap k , it is enough to find the first $\varphi(k)/2$ patterns. The remaining $\varphi(k)/2$ patterns are found by the relation mentioned above.

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