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"Various Reflexivities in Locally Convex Spaces and the Dual, Equipped with Different Topologies".

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Abstract: *In this paper we have considered various reflexivities, namely, polar semi-reflexivity, polar reflexivity, semi-reflexivity, reflexivity, inductive semi-reflexivity, inductive reflexivity, B-semireflexivity, B-reflexivity in locally convex spaces. We have investigated interrelationship (implication) between these reflexivities. Assuming that a locally convex space holds a type of reflexivity (out of the eight types), we have investigated whether or not, the dual E', equipped with the strong topology $\tau_b(E)$ or τ^* hold some of the eight types of reflexivity.*

Keywords: *strong dual, polar reflexive, reflexive, inductively reflexive, B-reflexive, bornological.*

I. Introduction

Throughout this paper, $E[\tau]$ will represent a locally convex topological vector space (abbreviated as locally convex space) over K (real or complex field).

For a locally convex space $E[\tau]$, the dual is denoted as E'. The strong dual of $E[\tau]$ is $E'[\tau_b(E)]$ and the bidual of $E[\tau]$ is $E'' = (E'[\tau_b(E)])'$. If $E''= E$, then $E[\tau]$ is called semi-reflexive. A semi-reflexive locally convex space $E[\tau]$ is called reflexive provided $\tau = \tau_b(E')$. Recall that a locally convex space is reflexive if and only if it is semi-reflexive and quasi-barreled. However, a reflexive locally convex space is always barreled. Semi-reflexivity and reflexivity are widely discussed in [1], [2]; and also, in [3, 4] on so called hereditary reflexivity.

Let τ^o be the topology on E' of uniform convergence on τ-precompact subsets of E and τ^{oo} be the topology on $(E'[\tau^o])'$ of uniform convergence on τ^o-precompact subsets of E'. Note that $\tau^o \leq \tau_b(E)$. If $(E'[\tau^o])' = E$, then $E[\tau]$ is called polar semi-reflexive, and polar reflexive if further $\tau = \tau^{oo}$. Polar semi-reflexive and polar reflexive locally convex spaces are considered in [5] as p-complete spaces and p-reflexive spaces, respectively (see [6]). We also note that polar reflexivity is the t-reflexivity of [7]. The class of polar semi-reflexive locally convex spaces coincides with the class of locally convex spaces in which every precompact subset is relatively compact. Every (F)-space is polar reflexive. For characterizations of polar semi-reflexivity and polar reflexivity, see [1], [5], [8], [9], [10], [11] and [12] (proposition-1.3).

Let τ^* be the strongest locally convex topology on E' for which all τ-equicontinuous subsets are bounded. Note that τ^* is always finer than the strong topology $\tau_b(E)$ and all the absolutely convex subsets of E' that absorbs all τ-equicontinuous subsets of E' form a basis of neighborhoods of 0 in $E'[\tau^*]$. If $(E'[\tau^*])'$ coincides with E, then $E[\tau]$ is called inductively semi-reflexive. Moreover, if $\tau = \tau^{**}$ i.e. $(\tau^*)^*$, then $E[\tau]$ is called inductively reflexive [13].

Recalled that a locally convex space $E[\tau]$ is said to have property-HC if every $f \in (E')^*$ which is bounded on equicontinuous sets is always in E. Inductive semi-reflexivity is equivalent to the property-HC (cf. [14]).

Note that for any set U in a locally convex space $E[\tau]$, $nU \subseteq U + U +\ldots+ U$ (sum of n terms); Further if U is convex, then $nU = U + U +\ldots+ U$. It is also recalled that a filter $\mathcal{F}$ on a locally convex space $E[\tau]$ is called almost Cauchy filter if for every neighborhood U of o, there is an integer n such that $nU \in \mathcal{F}$. Using this terminology it can be noted that a locally convex space $E[\tau]$ inductively semi-reflexive if and only if every almost Cauchy ultrafilter is $\tau_s(E')$-convergent [15].

Characterization of inductive semi-reflexivity in terms of extended Schauder decomposition is discussed by Webb, in [16]. Inductive (semi-) reflexivity is also discussed in [17] and [18] for three-space property.

let $\mathcal{R}$ be the class of all the absolutely convex bounded subset B of the dual E' whose span space E'_B is a reflexive Banach space with B as unit ball; such sets B are called reflective sets. Let τ_r be the topology, called reflective topology, on E of uniform convergence over the smallest saturated class of sets generated by $\mathcal{R}$. Note that if B is a finite set in E', then $B \in \mathcal{R}$ and so $\tau_s(E') \leq \tau_r$. On the other hand, if B is any absolutely convex bounded subset B of the dual E′ whose span space E'_B is a reflexive Banach space with B as unit ball, then $E \subseteq (E'_B)'$ and B is $\tau_s((E'_B)')$-relatively compact and so it is $\tau_s(E)$-relatively compact. Therefore, $\tau_r \leq \tau_k(E')$. Hence $\tau_s(E') \leq \tau_r \leq \tau_k(E')$. In particular, if E[τ] is barreled, then $\tau_s(E') \leq \tau_r \leq \tau$. A locally convex space E[τ] is said to be B-semireflexive if it is barreled and $E = \tilde{E}[\tilde{\tau}_r]$ (completion of $E[\tau_r]$). Further, if $\tau = \tau_r$, then E[τ] is called B-reflexive [19].

For many questions in analysis of locally convex spaces it is of crucial importance to know how the dual space behaves, when equipped with different topologies (viz $\tau_b(E)$, τ^*, τ^o). In particular, there are examples with the property that the dual of E′ (equipped with $\tau_b(E)$ or τ^*, τ^o) coincides with the original space, a condition for so called reflexivities for E[τ]. One of the main objectives of this paper is to give analysis and conclusions in this direction.

In this chapter, the term "various reflexivities" means (to cover and within the eight types) polar semi-reflexivity, polar reflexivity, semi-reflexivity, reflexivity, inductive semi-reflexivity, inductive reflexivity, B-semireflexivity, and B-reflexivity.

By "A type of reflexivity" we mean (any) one of these eight types of reflexivity. As for notation and terminology we generally follow Köthe [1].

We are required to note that a locally convex space E[τ] is called reinforced regular if every $\tau_s(E',(E'[\tau^*])')$-bounded set in $(E'[\tau^*])'$ is contained in $\tau_s(E', (E'[\tau^*])')$-closure of some bounded set in E[τ].

It is also required to mention the following well known results (cf.[13] and [19]), useful throughout this paper:

1.1 Theorem: A locally convex space is inductively reflexive if and only if it is inductively semi-reflexive and bornological.

1.2 Theorem: A locally convex space is inductively semi-reflexive if and only if it is semi-reflexive and reinforced regular.

1.3 Theorem: Every B-semireflexive locally convex space is a complete reflexive space.

1.4 Theorem: A reflexive locally convex space E[τ] is B-semireflexive if and only if $E'[\tau_b(E)]$ is bornological.

II. Interrelationships between Various Reflexivitiesies

In this section we present an approach to the study of interrelationships (implications) between various reflexivities. We start with the following two known lemmas which we have discussed in [20]; proofs are given for completeness:

2.1 Lemma: If a locally convex space E[τ] is inductively reflexive, then it is B-semireflexive.

Proof: Let E[τ] be inductively reflexive. From $(E'[\tau^*])' = E$ we have $\tau^* \leq \tau_k(E)$. But we always have $\tau_k(E) \leq \tau_b(E) \leq \tau^*$. Therefore $\tau_b(E) = \tau^*$ on E′. In particular, E[τ] is semi-reflexive. But E[τ] is bornological and so quasi-barrelled. Hence E[τ] is reflexive. It is clear that $E'[\tau^*]$ is bornological. So the strong dual $E'[\tau_b(E)] = E'[\tau^*]$ is bornological. Hence E[τ] is B-semireflexive (by theorem-1.4).

2.2 Lemma: If a locally convex space E[τ] is B-semireflexive, then it is reinforced regular.

Proof: Let E[τ] be B-semireflexive. It is barrelled and so every $\tau_b(E)$-bounded sequence in E′ is equicontinuous. Therefore, since the strong dual $E'[\tau_b(E)]$ is bornological, E[τ] is reinforced regular.

Now we assert the main interrelationship-theorem as follows:

2.3 Theorem: Let E[τ] be a locally convex space. Then

Inductively reflexive ⟹ Inductively semi-reflexive ⟸ B-semireflexive

⇓ ⇓ ⇓

B- Reflexive ⟹ B-semireflexive ⟹ Reflexive ⟹ Semi-reflexive

⇓ ⇓

Polar reflexive ⟹ Polar semi-reflexive.

Proof: If E[τ] is inductively reflexive, then it is B-semireflexive (Lemma- 2.1). A locally convex space is inductively semi-reflexive if and only if it is semi-reflexive and reinforeced regular (Theorem-1.2). Hence inductively semi-reflexive locally convex space is always semi-reflexive. B-semireflexive locally convex space is always reflexive (Theorem-1.3). Further, if E[τ] is B-semireflexive, then it is semi-reflexive. Also, by lemma-2.2, it is reinforced regular. Hence E[τ] is inductively semi-reflexive (Theorem-1.2). Semi-reflexivity implies polar semi-reflexivity and reflexivity implies polar reflexivity ([1]). Some of the implications are direct from definitions which completes the proof.

2.4 Counter examples: By illustrations as under, it is seen that the converse implications in the theorem are not true. It is also seen that other possible implications, like, inductively semi-reflexive ⇒ polar reflexive, polar reflexive ⇒ semi-reflexive, inductively semi-reflexive ⇒ reflexive, reflexive⇒ inductively semi-reflexive, are not true.

(i) Polar reflexive space which is not semi-reflexive: Any non-reflexive Banach space (viz. Banach space c_o) is a polar reflexive space which is not semi-reflexive.

Above example also works for-

(i)(a)) Polar semi-reflexive space which in not semi-reflexive, and

(i)(b)) Polar reflexive space which is not reflexive.

(ii) Semi-reflexive space which in not polar reflexive: The locally convex space $\ell^1[\tau_s(c_0)]$ is semi-reflexive ([21], theorem-2.8). The topology $(\tau_s(c_0))^o$ on the dual c_o is the normed topology. For this normed topology the set S= {(1, ½,…, 1/n ,0,0, …): n= 1, 2, …} is precompact which is not finite dimensional. Hence $\tau_s(c_0) \neq (\tau_s(c_0))^{oo}$. Therefore $\ell^1[\tau_s(c_0)]$ is not polar reflexive.

Another example: It is given in [7] (Ex.4.5: $\ell^2[\tau_s(\ell^2)]$ is semi-reflexive but not polar reflexive).

(iii) Semi-reflexive space which in not inductively semi-reflexive: The locally convex space $\ell^1[\tau_s(c_0)]$ is semi-reflexive. But it is not inductively semi-reflexive; otherwise it will be reinforced regular which is not true.

Another illustration is exhibited in [15] that there is a locally convex space which is complete and semi-reflexive but not inductively semi-reflexive.

(iv) Inductively semi-reflexive space which is not reflexive: The locally convex space $\ell^1[\tau_k(c_0)]$ is inductively semi-reflexive ([21], theorem 2.6). But it is not reflexive.

(v) Reflexive space which is not inductively semi-reflexive: An example of non-complete (M)-space space E is given in [22]. Since every (M)-space space is reflexive, so E is reflexive and non-complete. Since B-semireflexive space is always a complete reflexive space, hence this space E is not B-semireflexive. Further, a reflexive space is inductively semi-reflexive if and only if it is B-semireflexive. So the space E is not inductively semi-reflexive.

One more example of a reflexive locally convex space which is not complete is given in [3]. This locally convex space is reflexive but not B-semi-reflexive and so it is not inductively semi-reflexive.

Examples given in (v) also works for-

(v)(a)) Reflexive space which in not inductively reflexive:

(v)(b)) Reflexive space which is not B-semireflexive:

(vi) Inductively semi-reflexive space which is not B-semireflexive: In (iv), the locally convex space $\ell^1[\tau_k(c_0)]$ inductively semi-reflexive but it is not reflexive and so it is not B-semireflexive.

(vii) Inductively reflexive space which is not B-reflexive: Raman illustrated, in [19], (section-28(b)), a locally convex space H[τ] which is a reflexive (DF)-space and its strong dual $H'[\tau_b(E)]$ is a reflexive (F)-space but not B-reflexive. We observe that H[τ] and $H'[\tau_b(E)]$ are both Inductively reflexive (by [13], corollary-3.6). The locally convex space $H'[\tau_b(E)]$ inductively reflexive but not B-reflexive.

(viii) Polar semi-reflexive space which in not polar reflexive: In (ii), the locally convex space $\ell^1[\tau_s(c_0)]$ is semi-reflexive and so polar semi-reflexive. But it is not polar reflexive.

(ix) Semi-reflexive space which in not reflexive: The locally convex space $\ell^2[\tau_s(\ell^2)]$ is semi-reflexive ([21], corollary-2.5). But it not barreled and so it is not reflexive.

Second example: Let H_1 be the linear subspace of $\varphi\omega \oplus \omega\varphi$ consisting of all (x , x) with $x \in \varphi\omega \cap \omega\varphi = \varphi$ and H_2 the linear subspace of $\omega\varphi \oplus \varphi\omega$ consists of all (x , -x), $x \in \varphi$, (cf [1], §13,6). It is obtained that H_2 is a semi-reflexive locally convex space which is not reflexive.

(x) Inductively semi-reflexive space which in not inductively reflexive: In (iv), the locally convex space $\ell^1[\tau_k(c_0)]$ is inductively semi-reflexive but not reflexive and so it is not inductively reflexive.

(xi) B-semireflexive space which in not B-reflexive: From (vii) we see that the locally convex space $H'[\tau_b(E)]$ inductively reflexive (and so it is B-semireflexive) but not B-reflexive.

2.5 Remark: We do not know whether B-semireflexivity or B-reflexivity imply inductive reflexivity.

III. Dual E′, Equipped with the Strong Topology $\tau_b(E)$ and Various Reflexivity

The fact that a Banach space is reflexive if and only if its dual is reflexive curiosities the mind to think over the question– whether this fact can be generalized for locally convex spaces? Works of Mathematicians gave some short of answers like-(i) If a locally convex E[τ] is reflexive, the strong dual $E'[\tau_b(E)]$ is also reflexive; (ii) An (F)-space is reflexive if and only if its strong dual is reflexive. In this direction and even more- to see whether the above fact can be generalized for other types of reflexivity (out of the eight types) we deal with the problem of finding the truth value(s) of the statement(s): "If a locally convex space $E[\tau]$ holds a type of reflexivity, then the strong dual $E'[\tau_b(E)]$ (or $E'[\tau^*]$) holds some of the eight types of reflexivity". We consider it to discuss in the following manner:

(A) We consider the eight types of reflexivity and these eight types are, independently, allotted to the locally convex space $E[\tau]$ i.e., assume that $E[\tau]$ holds a type of reflexivity.

(B) The dual E' is equipped with the strong topology $\tau_b(E)$ or τ^*. We will justify if the dual space E', equipped with $\tau_b(E)$ or τ^*, can hold some of these reflexivities.

From (A) we have 64 cases of the above mentioned problem to be discussed for each of the two Cases of (B).

We start with the following:

3.1 Theorem: For a locally convex space $E[\tau]$, the strong dual $E'[\tau_b(E)]$ is polar semi-reflexive if and only if every precompact set is relatively compact in $E'[\tau_b(E)]$.

Proof: If $E'[\tau_b(E)]$ is polar semi-reflexive, then every $\tau_b(E)$-precompact set A is $\tau_s(E'')$- relatively compact. So $\tau_s(E'')$-closure of A is $\tau_s(E'')$- compact and so $\tau_b(E)$-complete. Hence A is $\tau_b(E)$-relatively compact.

Conversely, if every precompact set in $E'[\tau_b(E)]$ is relatively compact then it is also $\tau_s(E'')$- relatively compact. Therefore, $(\tau_b(E))^o \leq \tau_k(E')$ on E'' which implies that $(E''[(\tau_b(E))^o])' = (E''[\tau_k(E')])' = E'$. It means $E'[\tau_b(E)]$ is polar semi-reflexive.

We have established the following theorem in [6]:

3.2 Theorem: If $E[\tau]$ is polar reflexive, then its strong dual $E'[\tau_b(E)]$ is polar semi reflexive.

We discuss inductive semi-reflexivity for the strong dual in the following:

3.3 Theorem: If $E[\tau]$ is semi-reflexive, then the strong dual $E'[\tau_b(E)]$ is inductively semi-reflexive if and only if $\tau_k(E')$ is bornological topology on E.

Proof: Let $E[\tau]$ be semi-reflexive and $\tau_k(E')$ is bornological topology on E. We have $(E'[\tau_b(E)])' = E$. Further, since $\tau_k(E')$ is bornological topology on E, we have $\tau_k(E') = (\tau_b(E))^*$. Consider the locally convex space $E'[\tau_b(E)]$. Its dual is E and we have $(E[(\tau_b(E))^*])' = (E[\tau_k(E')])' = E'$. Hence $E'[\tau_b(E)]$ is inductively semi reflexive.

Conversely, if $E[\tau]$ is semi-reflexive and the strong dual $E'[\tau_b(E)]$ is inductively semi reflexive, then $E'[\tau_b(E)])' = E$ and $(E[(\tau_b(E))^*])' = E'$. It implies that $\tau_k(E') = (\tau_b(E))^*$.

Using the fact that for a locally convex space $E[\tau]$, every locally bounded linear functional on E is continuous if and only if $E'[\tau_k(E)]$ is bornological, this theorem can be stated as follows-

3.4 Corollary: If $E[\tau]$ is semi-reflexive, then the strong dual $E'[\tau_b(E)]$ is inductively semi reflexive if and only if every locally bounded linear functional on E is continuous.

From the fact that inductively semi reflexive locally convex space is always semi-reflexive, we obtain:

3.5 Corollary: If $E[\tau]$ is inductively semi reflexive, then the strong dual $E'[\tau_b(E)]$ is inductively semi- reflexive if and only if $\tau_k(E')$ is bornological topology on E.

Another result is established in the following:

3.6 Theorem: The strong dual of an inductively reflexive locally convex space is inductively reflexive.

Proof: If a locally convex space $E[\tau]$ is inductively reflexive, then $(E'[\tau^*])' = E$ and $\tau = \tau^{**}$. From $(E'[\tau^*])' = E$ we obtain that, on the dual E', $\tau_k(E) = \tau_b(E) = \tau^*$. Now consider the locally convex space $E'[\tau_b(E)]$. Its dual is E. We have $(E[(\tau_b(E))^*])' = (E[\tau^{**}])' = (E[\tau])' = E'$. It means $E'[\tau_b(E)]$ is inductively semi-reflexive. Again, since $E[\tau]$ is inductively reflexive, both $E[\tau]$ and $E'[\tau_b(E)]$ are bornological. Thus $E'[\tau_b(E)]$ is inductively semi-reflexive as well as bornological. Hence (by theorem-1.1) $E'[\tau_b(E)]$ is inductively reflexive.

3.7 Corollary: If a locally convex space $E[\tau]$ is inductively reflexive, then the dual $E'[\tau^*]$ is also inductively reflexive.

Proof: In the proof of the theorem we see that $\tau_b(E) = \tau^*$. So $E'[\tau^*] = E'[\tau_b(E)]$ which is inductively reflexive.

3.8 Corollary: If $E[\tau]$ is inductively reflexive, the strong dual $E'[\tau_b(E)]$ is always B-semireflexive.

Proof: Follows from above theorem and the fact that inductively reflexive locally convex space is always B-semireflexive.

Note that if $E[\tau]$ is B-semireflexive, then it is reflexive and therefore, the strong dual $E'[\tau_b(E)]$ is also reflexive. Further, we have

3.9 Theorem: If a locally convex space E[τ] is B-semireflexive, then the dual E′[τ*] is reflexive.
Proof: E[τ] is B-semireflexive, then E′[τ_b(E)] is bornological. We have τ_b(E) = τ*. Again, since E[τ] is reflexive, the strong dual E′[τ_b(E)] is reflexive. Using τ_b(E) = τ* we obtain that E′[τ*] is reflexive.

3.10 Theorem: If a locally convex space E[τ] is B-semireflexive, then the strong dual E′ [τ_b(E)] is B-semi-reflexive if and only if E[τ] is bornological.
Proof: Given that E[τ] is B-semireflexive. So it is reflexive i.e. (E′ [τ_b(E)])′ = E and τ = τ_b(E′). Its strong dual E′ [τ_b(E)] is also reflexive. If E′[τ_b(E)] is B-semi-reflexive, then its strong dual is bornological (by theorem-1.4). But we observe that the strong dual of E′ [τ_b(E)] is E[τ]. Hence E[τ] is bornological.
Conversely, Let E[τ] be bornological. Again we note that the strong dual of E′[τ_b(E)] is E[τ] which is bornological and the space E′[τ_b(E)] is reflexive. Hence E′ [τ_b(E)] is B-semireflexive.
We do not know whether a B-semireflexive locally convex space is always bornological.
Now we consider the following examples on various reflexivities in E′ [τ_b(E)]:

3.11 Example: Consider a non-reflexive Banach space E, it is always polar reflexive (and so polar semi-reflexive). Its strong dual is always a Banach space which is not semi-reflexive (and so none of - inductively semi-reflexive, reflexive, B-semireflexive, B-reflexive, inductively reflexive).

3.12 Example: The locally convex space E[τ}= ℓ^1[τ_k(c_0)] is inductively semi-reflexive (and so also semi-reflexive and polar semi-reflexive). Its strong dual E′[τ_b(E)] is the Banach space c_0 which is not semi-reflexive (and so none of- inductively semi-reflexive, reflexive, B-semireflexive, B-reflexive, inductively reflexive).

3.13 Example: As in the counter example-2.4-(vii) the locally convex space H[τ] inductively reflexive (and so also B-semireflexive, inductively semi-reflexive, reflexive, semi-reflexive, polar reflexive, polar semi-reflexive) but its strong dual H′[τ_b(E)] is not B-reflexive.

3.14 Example: There are (M)-spaces which are not bornological (for existence, see [11]). Such spaces are reflexive but their strong duals can't be inductively semi-reflexive (and so also can't be indctively reflexive, B-semireflexive, or B-reflexive)

3.15 Remark: First we note that if E[τ] is reflexive (and so if it is B-semireflexive or inductively reflexive), then strong dual E′[τ_b(E)] is also reflexive (and so also semi-reflexive, polar reflexive, polar semi-reflexive). Secondly, if E[τ] is inductively reflexive, then E′[τ_b(E)] is also inductively reflexive (and so also B-semi-reflexive, inductively semi-reflexive, reflexive, semi-reflexive. polar reflexive, polar semi-reflexive).

IV. Dual E′ , Equipped with the Topology τ* and Various Reflexivity

In this section, we consider to discuss different cases of various reflexivities in the dual E′[τ*].
4.1 Remark: By corollary-3.7, if E[τ] is an inductively reflexive locally convex space, then the dual E′[τ*] is inductively reflexive (and so also Inductively semi-reflexive, B-semireflexive, reflexive, semi-reflexive, polar reflexive, and polar semi-reflexive).

4.2 Remark: By theorem-3.9, If E[τ] is B-semireflexive (and so if it is B-reflexive, Inductively reflexive), then E′[τ*] is reflexive (and so also, semi-reflexive, polar reflexive, polar semi-reflexive).

4.3 Example: In the example-3.13 we have an inductively reflexive H[τ]. Its strong dual H′[τ_b(H)] is not B-reflexive. The strong dual H′[τ_b(H)] is bornological so τ_b(H) = τ* and so H′[τ*] is not B-reflexive.

4.4 Example: As in example-3.12, the locally convex space ℓ^1[τ_k(c_0)] is inductively semi-reflexive(and so semi-reflexive and polar semi-reflexive). Its strong dual c_0 [τ_b(ℓ^1)] is the Banach space c_0 which is not semi-reflexive. We have (τ_k(c_0))* = τ_b(ℓ^1). Hence the dual c_0[(τ_k(c_0))*] = c_0[τ_b(ℓ^1)] is not semi-reflexive (and so none of- inductively semi-reflexive, reflexive, B-semireflexive, B-reflexive, inductively reflexive).

We summarize the observations on the implication: E[τ] holds a type of reflexivity ⟹E′[τ_b(E)] holds some types of reflexivity, discussed in section-3 and the observations on the implication: E[τ] holds a type of reflexivity ⟹ E′[τ*] holds some types of reflexivity, discussed in section-4, respectively, in table-1 and table-2, using the following notations (for E[τ] and the duals E′[τ_b(E)] and E′[τ*]) :

I : Polar semi-reflexive,	II : Polar reflexive,
III : Semi-reflexive,	IV : Reflexive,
V : Inductively semi-reflexive,	VI : Inductively reflexive,

VII : B-semireflexive, VIII : B-reflexive.

These notations are called as states.

Observations on the implication: $E[\tau]$ **holds a type of reflexivity** $\Longrightarrow$ $E'[\tau_b(E)]$ **holds some types of reflexivity:**

$E[\tau]$									
	VIII	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	*	*	*	*
	VII	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	*	*	*	$\nRightarrow$
	VI	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\nRightarrow$
	V	*	*	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$
	IV	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$
	III	*	*	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$
	II	$\Rightarrow$	*	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$
	I	*	*	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$
		I	II	III	IV	V	VI	VII	VIII

$E'[\tau_b(E)]$ **(Table-1)**

$\Rightarrow$:State of $E[\tau]$ implies the state of $E'[\tau_b(E)]$;

$\nRightarrow$:State of $E[\tau]$ does not imply the state of $E'[\tau_b(E)\}$,

* : Not decided.

Observations on the implication: $E[\tau]$ **holds a type of reflexivity** $\Longrightarrow$ $E'[\tau^*]$ **holds some types of reflexivity:**

$E[\tau]$									
	VIII	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	*	*	*	*
	VII	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	*	*	*	$\nRightarrow$
	VI	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\nRightarrow$
	V	*	*	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$
	IV	*	*	*	*	*	*	*	$\nRightarrow$
	III	*	*	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$
	II	*	*	*	*	*	*	*	$\nRightarrow$
	I	*	*	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$	$\nRightarrow$
		I	II	III	IV	V	VI	VII	VIII

$E'[\tau^*]$ **(Table-2)**

$\Rightarrow$: State of $E[\tau]$ implies the state of $E'[\tau^*]$,

$\nRightarrow$: State of $E[\tau]$ does not imply the state of $E'[\tau^*]$,

* : Not decided.

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