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Restrained Double Domination in the Lexicographic Product of Graphs

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Abstract: *A double dominating set of G is a restrained double dominating set of G if for each $x \epsilon V(G) \setminus S$ there exists $y \epsilon V(G) \setminus S$ such that $xy \epsilon E(G)$. This paper, characterized the restrained double dominating sets in the lexicographic product of two graphs and determine sharp bounds for the restrained double domination numbers of these graphs. In particular, it shows that if G and H are connected graphs, then, $\gamma_{rx2}(G[H]) \leq min\{2\gamma_t(G), \gamma(G)\gamma_{rx2}(H)\}$, where $|G| \geq 2$ and $|H| \geq 3$, with γ, γ_t and γ_{rx2} are, respectively, the domination, total domination, and restrained double domination parameters.*

Keywords: *domination, lexicographic, restrained double domination, total domination*

I. Introduction

Let $G = (V(G), E(G))$ be a graph. For any vertex $x \epsilon V(G)$, the *open neighborhood of* x is the set $N_G(x) = \{y \epsilon V(G) : xy \epsilon E(G)\}$ and the *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \cup_{x\epsilon X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A subset S of $V(G)$ is *dominating set* in G if $N_G[S] = S \cup N_G(S) = V(G)$ where $N_G(S) = \{v \epsilon V(G) : xv \epsilon E(G)$ for some $x \epsilon S\}$. The minimum cardinality of a dominating set in G, denoted by $\gamma(G)$, is the *domination number* of G. A subset S of $V(G)$ where G is a graph with no isolated vertices is a *double dominating* set of G if for each $x \epsilon V(G)$, $|N_G[x] \cap S| \geq 2$. The *double domination number* of G, denoted by $\gamma_{x2}(G)$, is the minimum cardinality of a double dominating set of G. A double dominating set of G with cardinality $\gamma_{x2}(G)$ is called a γ_{x2}-set. Furthermore, a double dominating set of G is a *restrained double dominating set* of G if for each $x \epsilon V(G) \setminus S$, there exists $y \epsilon V(G) \setminus S$ such that $xy \epsilon E(G)$. The smallest cardinality of a restrained double dominating set of G is called the *restrained double domination number* of G and is denoted by $\gamma_{rx2}(G)$. A restrained double dominating set of G with cardinality $\gamma_{rx2}(G)$ is called a γ_{rx2}-set.

Double dominating set and double domination number were first defined and introduced by F. Harary and T. W. Haynes in [1] as cited in [2]. They also established the Nordhaus-Gaddum inequalities for double domination. The concept of restrained domination was introduced by Telle in [3] as cited in [4] when they determine the best possible upper and lower bounds for $\gamma_r(G)$. R. Kala and T. R. Nirmala Vasantha[5] initiated the study of restrained double domination and obtained some bounds for $\gamma_{rx2}(G)$ and characterized the graphs obtaining those bounds. Cuivillas and Canoy[6] characterized the double dominating sets in the join, corona, and lexicographic product of two graphs. They also determined sharp bounds for the double domination numbers of these graphs.

II. Preliminary

The *lexicographic product* or *composition* $G[H]$ of two graphs G and H is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the conditions: $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in V(H)$.

Given $C \subseteq V(G) \times V(H)$, the G-projection C_G and H-projection C_H of C are, respectively,

$C_G = \{x \in V(G) : (x, y) \in C$ for some $y \in V(H)\}$

and

$C_H = \{y \in V(H) : (x, y) \in C$ for some $x \in V(G)\}$.

Now, let C be a non-empty subset of $V(G) \times V(H)$. For each $x \in S = C_G$, let $T_x = \{a \epsilon V(H): (x, a) \in C$. Then

$$C = \bigcup_{x \in S} [\{x\} \times T_x].$$

Theorem 2.1 *Let G and H be any connected non-trivial graphs. A non-empty subset $C = \cup_{x\in S}[\{x\} \times T_x$ of VGH is a double dominating set of GH if and only if S is a dominating set of G and satisfies each of the following:*

(a) For each $x \in V(G) \setminus S$ such that $|N_G(x) \cap S| = 1$, $|T_y| \geq 2$ for $y \in N_G(x) \cap S$.
(b) T_x is a double dominating set of H for each $x \in S \setminus N(S)$.
(c) For each $z \in S$ with $|N_G(z) \cap S| = 1$, either T_z is a dominating set of H or $|T_w| \geq 2$ for $w \in N_G(z) \cap S$.

Proof: Suppose S is a dominating set of G. Let $x \in V(G) \setminus S$ with $|N_G(x) \cap S| = 1$, say $N_G(x) \cap S = \{y\}$. Pick any $a \in V(H)$. Then $(x, a) \notin C$. Since S is a double dominating set, there exists $(y, b), (y, c) \in C \cap N_{G[H]}((x, a))$. Thus, $|T_y| \geq 2$. Next, let $z \in S$ with $|N_G(z) \cap S| = 1$. Let $\{w\} = N_G(z) \cap S$. If T_z is a dominating set, then we are done. Suppose T_z is not a dominating set. Then there exists $a \in V(H) \setminus T_z$ such that $ab \notin E(H)$ for all $b \in T_z$. Since $(z, a) \notin C$ and C is a double dominating set of $G[H]$, there exists $p, q \in T_w\, (p \neq q)$ such that $(w, p), (w, q) \notin N_{G[H]}((z, a))$. This implies that $|T_w| \geq 2$. Finally, let $x \in S \setminus N(S)$ and let $c \in V(H) \setminus T_x$. Since C is a double dominating set, there exists $a, b \in T_x\, (a \neq b)$ such that $(x, a), (x, b) \notin N_G((x, c))$. This implies that $a, b \in N_G(c)$. Moreover, since C is a total dominating set of $G[H]$, T_x is a total dominating set of H. Thus, T_x is a double dominating set of H.

For the converse, suppose S is a dominating set of G satisfying (a), (b), and (c). Let $(x, a) \in V(G[H])$. Consider the following cases:

Case 1: $x \in S$

If $x \in S \setminus N(S)$, then T_x is a double dominating set of H by (b). If $a \in T_x$, then there exists $b \in T_x \setminus \{a\}$ such that $ab \in E(H)$. Hence, there exists $(x, b) \in C$ such that $(x, a) \in C \cap N_{G[H]}((x, b))$. If $a \notin T_x$, then there exists $c, d \in T_x (c \neq d)$ such that $(x, a) \in N_{G[H]}((x, c)) \cap N_{G[H]}((x, d))$. Suppose $N_G(x) \cap S = \{w\}$. Pick any $p \in T_w$. Then $(x, a) \in C \cap N_{G[H]}((w, p))$. Suppose $a \notin T_x$. If T_x is a dominating set, then there exists $b \in T_x$ such that $ab \in E(H)$. Hence $(x, a) \in C \cap N_{G[H]}((w, p)) \cap N_{G[H]}((x, b))$. If T_x is not a dominating set, then $|T_w| \geq 2$ by (c). This implies that there exists $q \in T_w \setminus \{p\}$. Therefore, $(x, a) \in C \cap N_{G[H]}(\{(w, p), (w, q)\})$.

Case 2: $x \notin S$

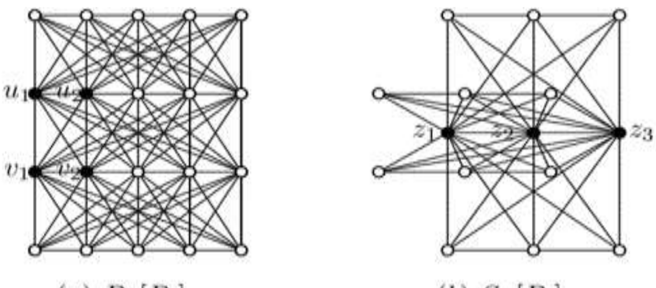
Suppose $x \notin S$. Then $(x, a) \notin C$. From our assumption that (a) holds, it follows that $|T_y| \geq 2$ if $N_G(x) \cap S = \{y\}$. Pick $b, c \in T_y$, where $b \neq c$. Then $(x, a) \in C \cap N_{G[H]}(\{(y, b), (y, c)\})$. Suppose $|N_G(x) \cap S| \geq 2$. Choose any $y, z \in N_G(x) \cap S$ where $y \neq z$. Pick any $s \in T_y$ and $t \in T_z$. Then $(x, a) \in C \cap N_{G[H]}(\{(y, s), (y, t)\})$. Accordingly, C is a double dominating set of G. ■

Corollary 2.2 *Let G and H be a connected non-trivial graphs of orders $n \geq 2$ and $m \geq 2$, respectively. Then,*

$$\gamma_{x2}(G[H]) \leq \min\{2\gamma_t(G), \gamma(G)\gamma_{x2}(H)\}.$$

Proof: Let S_1 and S_2 be $\gamma -$ and $\gamma_t - sets$ of G and let D_1 be a $\gamma_{x2} - set$ of H and $D_2 = \{a, b\}$, where $a, b \in V(H)$. Set $C_1 = \cup_{x \in S_1}[\{x\} \times D_1] = S_1 \times D_1$ and $C_2 = \cup_{x \in S_2}[\{x\} \times D_2] = S_2 \times D_2$. By Theorem 2.1, C_1 and C_2 are double dominating sets of $G[H]$. Hence, $\gamma_{x2}(G[H]) \leq |C_1| = \gamma(G)\gamma_{x2}(H)$ and $\gamma_{x2}(G[H]) \leq |C_2| = 2\gamma_t(G)$. Thus, $\gamma_{x2}(G[H]) \leq \min\{2\gamma_t(G), \gamma(G)\gamma_{x2}(H)\}$. ■

Example 2.3 Consider the graphs $P_4[P_5]$ and $S_4[P_3]$ in Fig. 1. In Fig. 1(a), $\gamma_{x2}(P_4[P_5]) = 2\gamma_t(P_4) = 2(2) = 4 < \gamma(P_4)\gamma_{x2}(P_5) = 2(4) = 8$, while, in Fig. 1(b), $\gamma_{x2}(S_4[P_3]) = \gamma(S_4)\gamma_{x2}(P_3) = 1(3) = 3 < 2\gamma_t(S_4) = 2(2) = 4$. These graphs in Fig. 1 show that the bounds given in Corollary 2.2 are sharp.



(a) $P_4[P_5]$ (b) $S_4[P_3]$

Figure 1. Lexicographic product of different graphs resulting to different γ_{x2}.

III. Lexicographic Product Of Graphs

Theorem 2.1*Let* G *and* H *be any connected non-trivial graphs. A non-empty subset* $C = \cup_{x\in S}[\{x\} \times T_x]$ *of* $V(GH)$ *is a restrained double dominating set of* $G[H]$ *if and only if* S *is a dominating set of* G *and satisfies each of the following:*

(i) *For each* $x \in V(G) \setminus S$ *such that* $|N_G(x) \cap S| = 1, |T_y| \geq 2$ *for* $y \in N_G(x) \cap S$;

(ii) T_x *is a double dominating set of* H *for each* $x \in S \setminus N(S)$;

(iii) *For each* $z \in S$ *with* $|N_G(z) \cap S| = 1$, T_z *is a restrained dominating set of* H *or* T_z *is a dominating set of* H *and either* $|V(H) \setminus T_w| \geq 1$ *or* $N_G(z) \cap (V(G \setminus S) \neq \emptyset$ *or* $|T_w| \geq 2$ *and* $\langle V(H) \setminus T_z\rangle$ *has no isolated vertex, or* $|T_w| \geq 2$ *and either* $|V(H) \setminus T_w| \geq 1$, *or* $N_G(z) \cap (V(G) \setminus S) \neq \emptyset$, *where* $w \in N_G(z) \cap S$; *and*

(iv) *For each* $z \in S$ *with* $|N_G(z) \cap S| \geq 2$, *either* $\langle V(H) \setminus T_z\rangle$ *has no isolated vertex or* $T_v \neq V(H)$ *for some* $v \in N_G(z) \cap S$.

Proof: Suppose $C = \cup_{x\in S}[\{x\} \times T_x]$ is a restrained double dominating set of $G[H]$. Then S is a dominating set of G. Let $x \in V(G) \setminus S$ with $|N_G(x) \cap S| = 1$, say $N_G(x) \cap S = \{y\}$. Pick any $a \in V(H)$. Then $(x, a) \notin C$. Since C is a double dominating set, there exists $(y, b), (y, c) \in C \cap N_{G[H]}((x, a))$. Thus, $|T_y| \geq 2$.

Next, let $x \in S \setminus N(S)$ and let $c \in V(H)$. Since C is a double dominating set of $G[H]$, $|N_{G[H]}[(x, c)] \cap C| \geq 2$. It follows that $|N_{H[C]} \cap T_x| \geq 2$. Therefore T_x is a double dominating set of H.

Let $z \in S$ with $|N_G(z) \cap S| = 1$, say $\{w\} = N_G(z) \cap S$. If T_z is a restrained dominating set, then we are done. Suppose T_z is not a restrained dominating set. If T_z is a dominating set, then there exists $p \notin T_z$ such that $p \notin N_H(q)$ for all $q \notin T_z$. Since C is a restrained dominating set, either $T_w \neq V(H)$ or $N_G(z) \cap (V(G) \setminus S) \neq \emptyset$. If T_z is not a dominating set, then there exists $a \notin T_z$ such that $a \notin N_H(b)$ for all $b \in T_z$. Since C is a double dominating set, $|T_w| \geq 2$. If $\langle V(H) \setminus T_z\rangle$ has an isolated vertex, then either $T_w \neq V(H)$ or $N_G(z) \cap (V(G) \setminus S) \neq \emptyset$.

For the converse, suppose S is a dominating set of G satisfying *(i)*, *(ii)*, *(iii)*, and *(iv)*. Let $(x, a) \in V(G[H])$. Consider the following cases:

Case 1: Suppose $x \in S$.

If $x \in S \setminus N(S)$, then T_x is a double dominating set of H by *(ii)*. Then , $|N_H[a] \cap T_x| \geq 2$. It follows that $|N_{G[H]}[(x, a)] \cap C| \geq 2$. Also, since G is a connected non-trivial graph, there exists $y \in V(G) \setminus S$ such that $(y, a)(x, a) \in E(G[H])$. Suppose $N_G(x) \cap S = \{w\}$. Pick any $p \in T_w$. If $a \in T_x$, then $(w, p) \in C \cap N_{G[H]}((x, a))$. Suppose $a \notin T_x$. If T_x is a restrained dominating set, then there exist $b \in T_x$ and $d \notin T_x$ such that $b, d \in N_H(a)$. This implies that $(w, p), (x, b) \in C \cap N_{G[H]}((x, a))$ and $(x, d) \in V(G[H] \setminus C) \cap N_{G[H]}((x, a))$. If T_x is not a restrained dominating set, then there exist $(y, t), (z, q) \in C \cap N_{G[H]}((x, a))(y = z = w$ or $y = w$ and $z = x)$ and $(v, s) \in V(G[H]) \setminus C) \cap N_{G[H]}((x, a))(z \in (V(G) \setminus S \cap N_G(x)$ or $z = w)$ by *(iii)*. Furthermore, if $|N_G(x) \cap S| \geq 2$, then $|N_{G[H]}[(x, a)] \cap C| \geq 2$. By *(iv)*, there exists $(u, k) \in V(G[H]) \setminus C) \cap N_{G[H]}((x, a))$.

Case 2: Suppose $x \notin S$.

Then $(x, a) \notin C$. If $|N_G(x) \cap S| \geq 2$, then $|N_{G[H]}((x, a)) \cap C| \geq 2$. If $|N_G(x) \cap S| = 1$, say $N_G(x) \cap S = \{y\}$, then $|T_y| \geq 2$ by *(i)*. Pick any $b, c \in T_y$, where $b \neq c$. Then $(y, b), (y, c) \in C \cap N_{G[H]}((x, a))$. Moreover, there exists $(x, s) \in (V(G[H]) \setminus C) \cap N_{G[H]}((x, a))$ since H is a non-trivial connected graph. Accordingly, C is a restrained double dominating set of G. ■

Corollary 3.2*Let* G *and* H *be a connected graphs of orders* $n \geq 2$ *and* $m \geq 3$, *respectively. Then,*

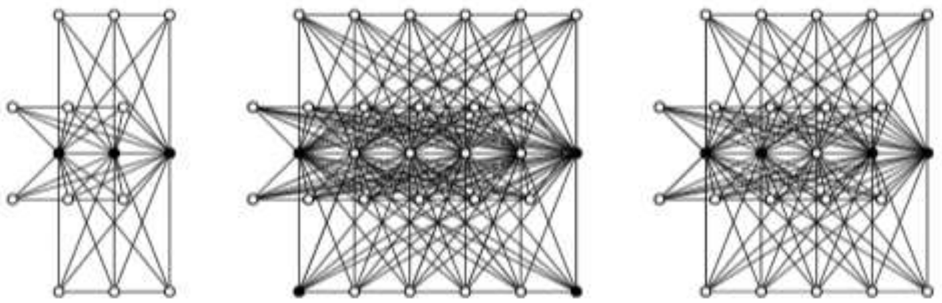
$$\gamma_{r\times 2}(G[H]) \leq min\{2\gamma_t(G), \gamma(G)\gamma_{r\times 2}(H)\}.$$

Proof: Let S_1 be $\gamma - set$ and S_2 be $\gamma_t - set$ of G and let D_1 be a $\gamma_{r\times 2} - set$ of H and $D_2 = \{a, b\}$, where $a, b \in V(H)$. Set $C_1 = \cup_{x\in S_1}[\{x\} \times D_1] = S_1 \times D_1$ and $C_2 = \cup_{x\in S_2}[\{x\} \times D_2] = S_2 \times D_2$. Then, by Theorem 3.1, C_1 and C_2 are restrained double dominating sets of $G[H]$. It follows that, $\gamma_{r\times 2}(G[H]) \leq |C_1| = \gamma(G)\gamma_{r\times 2}(H)$ and $\gamma_{r\times 2}(G[H]) \leq |C_2| = 2\gamma_t(G)$. Thus, $\gamma_{r\times 2}(G[H]) \leq min\{2\gamma_t(G), \gamma(G)\gamma_{r\times 2}(H)\}$.

This proves the assertion. ■

Example 3.3 Consider the graphs $K_{1,4}[P_3]$, $K_{1,4}[P_6]$, and $K_{1,4}[P_6]$ in Fig. 2. In Fig. 2(a), it can be seen that $\gamma_{r\times 2}(K_{1,4}[P_3]) = \gamma(K_{1,4})\gamma_{r\times 2}(P_3) = 1(3) = 3 < 2\gamma_t(K_{1,4}) = 2(2) = 4$. In Fig. 2(b), $\gamma_{r\times 2}(K_{1,4}[P_6]) = 2\gamma_t(K_{1,4}) = 2(2) = 4 < \gamma(K_{1,4})\gamma_{r\times 2}(P_6) = 1(5) = 5$, and in Fig.2(c),

$\gamma_{r\times2}(K_{1,4}[P_5]) = \gamma(K_{1,4})\gamma_{r\times2}(P_5) = 1(4) = 4 = 2(2) = 2\gamma_t(K_{1,4})$. Thus, the upper bound given in Corollary 3.2 are sharp.



(a) $K_{1,4}[P_3]$ (b) $K_{1,4}[P_6]$ (c) $K_{1,4}[P_5]$

Figure 2. The graphs of $K_{1,4}[P_3]$, $K_{1,4}[P_6]$,and $K_{1,4}[P_6]$.

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