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Integral Solutions of Ternary Quadratic Diophantine Equation $7(x^2+y^2)-13xy=27z^2$

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Abstract: *The ternary quadratic Diophantine equation given by* $7(x^2+y^2)-13xy=27z^2$ *is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.*
Keywords: *Ternary quadratic, integral solutions, polygonal numbers.*

I. Introduction

Ternary quadratic equations are rich in variety [1-3].For an extensive review of sizable literature and various problems, one may refer [1-20]. In this communication, we consider yet another interesting ternary quadratic equation $7(x^2+y^2)-13xy=27z^2$and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special polygonal numbers are presented.

Notations Used

- $t_{m,n}$- Polygonal number of rank 'n' with size 'm'
- P_n^m- Pyramidal number of rank 'n' with size 'm'
- $F_{m,n}$–Figurative number of rank 'n' with size 'm'

Methods of Analysis

The Quadratic Diophantine equation with three unknowns to be solved for its non zero distinct integral solutions is

$$7(x^2+y^2)-13xy=27z^2 \quad (1)$$

On substituting the linear transformations

$$x=u+v\,; \quad y=u-v \quad (2)$$

in (1), leads to

$$u^2=27(z^2-v^2) \quad (3)$$

Pattern I

Equation (3) can be written as

$$u^2+27v^2=27\,z^2 \quad (4)$$

Assume $z^2=A^2+27$ (5)

27 can be written as

$$27=(i\sqrt{27})(-i\sqrt{27}) \quad (6)$$

Substitute $(4),(5) in (3)$and applying the method of factorization, we get

$$u+i\sqrt{27}v=-54AB+i\sqrt{27}(A^2-27B^2) \quad (7)$$

Equating real and imaginary parts

$$u=-54\,AB\ ;\ \ v=A^2-27\,B^2 \quad (8)$$

Using $(8),(5) and (2)$ we obtain the integer solutions to (1)as presented below

$$\left.\begin{aligned} x(A,B)=x&=A^2-27\,B^2-54AB\\ y(A,B)=y&=-A^2+27B^2-54AB\\ z(A,B)=z&=A^2+27\,B^2\end{aligned}\right\}(9)$$

Properties

1. $z(1,B)-x(1,B)-108t_{3,B}\equiv 0$
2. $z(1,B(B+2))-x(1,B(B+2))-324P_B^3 \equiv 0$
3. $z(1,B(B+2)(B+3))-x(1,B(B+2)(B+3))-1296\,P_B^4 \equiv 0$
4. $z(1,B^2)-x(1,B^2)-108P_B^5 \equiv 0$
5. $z(1,B(B+1)(B+2))-x(1,B(B+1)(B+2))-648\,F_{4,B-4} \equiv 0$
6. $y(1,1)+z(1,1)\equiv 0$
7. $x(A,A)+y(A,A)$+ z(A,A) can be expressed as a sum of two squares
8. $x(1,2)+y(1,2)$ is a cube no
9. Each of the following expression represents a perfect number
   a. $y(1,1)$
   b. $z(1,1)$
10. Each of the following expression represents a Nasty number
    a. $\frac{1}{2}[x(1,1)-z(1,1)]$
    b. $\frac{1}{2}[x(1,1)+y(1,1)]$
    c. $y(0,1)+z(0,1)$
    d. $x(2,2)-y(2,2)+z(2,2)$
    e. $x(A,-A)+y(A,-A)+z(A,-A)$
11. Each of the following can be expressed as a perfect squares
    a. $y(1,3)+z(1,3)$
    b. $x(1,3)+y(1,3)$
    c. $x(2,1)-\ z(2,1)$

Pattern II

write(3) as

$$u^2+27v^2=27\,z^2*1 \qquad (10)$$

Assume

$$z^2=A^2+27\,B^2 \qquad (11)$$

Also 27 can be written as

$$1=\frac{(3+i\sqrt{27})(3-i\sqrt{27})}{36} \qquad (12)$$

Applying $(17),(16)in(15)$and employing the method of factorization, define

$$u+i\sqrt{27}v=\frac{1}{6}i\sqrt{27}(3A^2-81B^2-54AB)+\frac{1}{6}\,(-27A^2+729B^2-\ 162AB) \qquad (13)$$

Equating real and imaginary parts

$$\left.\begin{aligned} u&=\frac{1}{6}(-27A^2+729\,B^2-162AB)\\ v&=\frac{1}{6}(3A^2-81B^2-54AB)\end{aligned}\right\} \qquad (14)$$

Using $(14),(11)and(2)$ we obtain the integer solutions to (1)as presented below

$$\left.\begin{aligned} x(A,B)&=x=-4A^2+108B^2-36AB\\ y(A,B)&=y=-5A^2+135B^2-18AB\\ z(A,B)&=\ z=27\,B^2+A^2\end{aligned}\right\}(15)$$

Properties

1. $5z(A,1)-y(A,1)-20t_{3,A}\equiv 0(\,mod\ 8)$
2. $4z(A,1)-x(A,1)-16t_{3,A}\equiv 0\ (mod\ 36)$

3. $4y(A(A+1),1)-5x(A\,(A+1),1)-216t_{3,A}\equiv 0$
4. $4y(A(A+1)(A+2),1)-5x(A\,(A+1)(A+2),1)-648\,P_A^3\equiv 0$
5. $4y(A(A+1)(A+2)(A+3),1)-5x(A\,(A+1)(A+2)(A+3),1)-2592\,P_A^4\equiv 0$
6. $4y(A^2(A+1),1)-5x(A^2(A+1),1)-216\,P_A^5\equiv 0$
7. $4y(A(A+1)^2(A+2),1)-5x(A\,(A+1)^2(A+2),1)-1296F_{4,n-4}\equiv 0$
8. $x(1,1)+4z(1,1)$is a nasty number.
9. Each of the following expression represents a perfect square
   a. $[y(1,1)-5z(1,1)]-[x(1,1)-4z(1,1)]$
   b. $y(1,1)-x(1,1)-z(1,1)$
   c. $y(2,2)-\ x(2,2)-z(2,2)$
   d. $y(2,2)-x(2,2)+z(2,2)$
   e. $x(2,3)-y(2,3)+z(2,3)$
   f. $y(2,3)-x(2,3)-z(2,3)$
10. The following expression represents a perfect no
   a. $y(1,1)-5z(1,1)$
   b. $y(2,1)-\ x(2,1)-z(2,1)$
   c. $x(2,1)+z(2,1)-y(2,1)$

Pattern III

Equation (3)$can be written as$

$$\frac{u}{27(z+v)}=\frac{z-v}{u}=\frac{A}{B}\,,\qquad B\neq 0 \tag{16}$$

which is equivalent to the system of equations

$$Au-Bz+Bv=0$$
$$Bu-27Az-27Av=0$$

From which we get

$$\left.\begin{aligned} u&=54AB\\ v&=B^2-27A^2\\ z&=27A^2+B^2\end{aligned}\right\}(17)$$

Substituting (17)and (16)in (2), the corresponding non-zero distinct integral solutions of(1)are given by

$$\left.\begin{aligned} x(A,B)&=x=B^2-27A^2+54AB\\ y(A,B)&=y=27A^2-B^2+54AB\\ z(A,B)&=z=27A^2+B^2\end{aligned}\right\}(18)$$

Properties

1. $x(A,(A+1))+y(A,(A+1))-216t_{3,A}\equiv 0$
2. $x(A,A\,(A+1))+y(A,A(A+1))-216P_a^5\equiv 0$
3. $x\big(A,(A+1)(A+2)(A+3)\big)+y\big(A,(A+1)(A+2)(A+3)\big)-2592P_A^4\equiv 0$
4. $x(A,(A+1)(A+2))+y(A,(A+1)\,(A+2))-648P_A^3\equiv 0$
5. $y(A,1)-54\,t_{3,A}\equiv -1\ (mod\ \ 3)$
6. $z(A,1)-t_{56,A}\equiv 1(mod\ 55)$
7. $x\,(1,B)+z(1,B)-4t_{3,B}\equiv 0\ (mod\ 13)$
8. $x(A(A+1),(A+1)(A+2))-1296F_{4,n-4,}\equiv 0$
9. $y(A,1)+z(A,1)-108t_{3,A}\equiv 0$
10. $x(1,1)+y(1,1)$ is expressed as difference of two perfect squares
11. $x(1,1)+y(1,1)+z(1,1)$ can be expressed as a sum of two perfect squares
12. Each of the following expression represents a perfect number
   a. $y(3,1)-x(3,1)$
   b. $x(3,2)-y(3,2)+z(3,2)$
   c. $y(2,3)-\ z(2,3)$
13. Each of the following expression represents a Nasty number
   a. $(1,2)-y(3,3)$

b. $y(1,1)-x(1,1)-z(1,1)$
c. $y(2,2)-x(2,2)-z(2,2)$
d. $\frac{1}{3}\,z(3,3)$

It is observed that6, by rewriting (3)suitably, one may arrive at the following pattern of solutions to (1)

Pattern IV

$$\left.\begin{aligned} x(A,B)=x&=27\,B^2+54AB-A^2\\ y(A,B)=y&=A^2+54AB-27\,B^2\\ z(A,B)=z&=A^2+27\,B^2 \end{aligned}\right\}(19)$$

Properties

1. $z(A,B)-t_{56,A}\equiv 1(mod\ 55)$
2. $y(A,B)-4t_{3,A}\equiv -27\ (mod\ 53)$
3. $x(1,B)+z(1,B)-108\,t_{3,B}\equiv 0$
4. $x(A(A+1),1)+z(A(A+1),1)-108\,t_{3,A}\equiv 0\ (mod\ 13)$
5. $x(A(A+1)(A+2),1)+z(A\,(A+1)(A+2),1)-324\,P_A^3\equiv 0\ (mod\ 13)$
6. $x(A(A+1)(A+2)(A+3),1)+z(A\,(A+1)(A+2)(A+3),1)-1296\,P_A^4\equiv 0\ (mod\ 13)$
7. $x(A^2(A+1),1)+z(A^2(A+1),1)-108\,P_A^5\equiv 0\ (mod\ 13)$
8. $x(A(A+1)^2(A+2),1)+z(A\,(A+1)^2(A+2),1)-648\,F_{4,n-4}\equiv 0(mod\ 13)$
9. $y(1,2)+y(3,3)$ can be expressed as a sum of two squares
10. Each of the following expression represents a perfect number
    a. $y(1,1)$
    b. $z(1,1)$
11. Each of the following expression represents a Nasty number
    a. $x(1,1)-y(1,1)-z(1,1)$
    b. $x(2,2)-y(2,2)-z(2,2)$
    c. $x(1,1)+y(1,2)$
12. Each of the following can be expressed as a difference of two squares
    a. $z(3,3)$
    b. $y(2,2)$

II. Conclusion

In this paper, we have presented five different patterns of non- zero distinct integer solutions of ternary quadratic Diophantine equation $7(x^2+y^2)-13\,xy=27z^2$and relations between solutions and special numbers are also obtained. To conclude, one may search for other patterns of solutions and their corresponding properties.

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