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A Physical Interpretation of Riesz Potential

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Abstract: *In this paper we add a physical meaning to the Riesz potential by using Newtonian force fields due to a body and velocity fields of any fluid. To do this, we firstly give some properties of Newtonian force fields and correspondingly Newtonian potentials. Then, we introduce the Riesz potentials which is arised from fractional order* $(\alpha/2)$ *Poisson differential equations like Newton potentials and by using these notions we give a physical interpretation of Riesz potential. Finally, we cite the flow velocity of a fluid and add a physical meaning to Riesz potential by using velocity potential of a fluid that is irrotational in some simply connected region in the same way*

Keywords: *Vector Fields-Flows, Potentials, Laplace's Equations*

I. Introduction

While the theory of Newtonian potentials has various aspects, it is best introduced as a body of results on the properties of forces which are characterized by Newtons Law of Universal Gravitation. The magnitude of the force between two particles, one of mass m_1, situated at a point P, and one of mass m_2, situated at Q, is given by Newton's law as

$$F = \gamma\left(\frac{m_1 m_2}{r^2}\right)$$

where r is distance between P and Q. The constant of proportionality γ depends solely on the units used. Also we shall take $\gamma = 1$ and the components of a vector field due to a volume distribution situated at a point $Q(\xi,\eta,\zeta)$ with density σ (we shall assume that density is continuous) and volume V is given as follows

$$X = \int_V \frac{\sigma(\xi - x)}{r^2} dV \ , \ Y = \int_V \frac{\sigma(\eta - y)}{r^2} dV \ , \ Z = \int_V \frac{\sigma(\zeta - z)}{r^2} dV$$

for some point $P(x, y, z)$ [1].

II. Potentials

A particle of mass m, subject only to the force of a specific field (X,Y,Z) will move in accordance with Newton's second law of motion [3]:

$$\lambda\mathbf{F} = m\frac{d}{dt}\mathbf{v}$$

If we rearrange this expression, then we have the change in kinetic energy of particle as follows

$$E - E_0 = \int_{t_0}^{t} \mathbf{F}\mathbf{v}\,dt = \int_{P_0}^{P} \mathbf{F}d\mathbf{s} = W(P_0, P, C)$$

where $\mathbf{v}$, E and $d\mathbf{s}$ represent velocity, kinetic energy and elemental displacement of the particle. The quantity $W(P_0, P, C)$ is the work required to move the particle from point P_0 to P. This equation shows that the change in kinetic energy of the particle equals the work done by $\mathbf{F}$. In general, the work required to move the particle from P_0 to P differs depending on the path taken by the particle. But here we consider work functions independent of the paths taken by the particles which give us conservative fields that can be expressed by gradient of the work.

The potential ϕ of vector field $\mathbf{F}$ is defined as the work function or as its negative depending on the convention used. Kellogg [1] summarizes these conventions as follows:

i) If particles of like sign attract each other (e.g., gravity fields), then $\mathbf{F} = \nabla\phi$ and the potential equals the work done by the field.

ii) If particles of like sign repel each other (e.g., electrostatic fields), then $\mathbf{F} = -\nabla\phi$, and the potential equals the work done against the field by the particle. In the latter case, the potential ϕ is the potential energy of the particle; in the former case, ϕ is the negative of the particle's potential energy. In Newtonian fields, the potentials at $P(x,y,z)$ due to a particle which have mass m and a body which have volume V and density σ at some point $Q(\xi,\eta,\zeta)$ are

$$\phi = \frac{m}{r}$$

and

$$\phi = \iiint_V \frac{\sigma}{r}\,dV$$

respectively. In the Newtonian fields, the potentials of all the distributions studied satisfy Poisson's equation

$$\Delta\phi = -4\pi\sigma$$

at all points, even inside the mass distribution. In particular it satisfies Laplace's equation

$$\Delta\phi = 0$$

at all points outside of the mass.
If we take fractional Laplacian $(-\Delta)^{\alpha/2}$, we have

$$(-\Delta)^{\alpha/2}\phi(u) = -f(u)$$

The solution of this equation is

$$\phi = C_n^\alpha \int_{\mathbb{R}^n} \frac{f(v)}{|u-v|^{n-\alpha}}\,dv$$

which is called Riesz potential. Here C_n^α is a constant [2,4,5].

Here we firstly consider Newtonian and Riesz potentials together and interpret force field corresponding to them.To do this, we take $n=3$ and $0<\alpha<2$. Then consider

$$\Phi = U + CR = \iiint_V \frac{f(y)}{r}\,dy + C\iiint_V \frac{f(y)}{r^{3-\alpha}}\,dy$$

where U, R are Newtonian and Riesz potentials respectively and C is a constant. The force field corresponding to them will be as follows:

$$\mathbf{F} = \iiint_V \left[\frac{(x-y)}{r^3} + C_\alpha \frac{(x-y)}{r^{5-\alpha}}\right] f(y)\,dy$$

where C_α is a constant that arises after taking derivatives of Φ which depends on C and α . By thinking that force fields are inversely proportional to square of distance between body and any point of the free space,if we do rearrangement then we obtain

$$\mathbf{F} = \iiint_V \frac{(x-y)}{r}\frac{1}{r^2} f(y)\left[1 + \frac{C_\alpha}{r^{2-\alpha}}\right]dy$$

where $\frac{(x-y)}{r}$ is direction cosine and then

$$\sigma = f(y)\left[1 + \frac{C_\alpha}{r^{2-\alpha}}\right]$$

must be the density.

As we see, the density is changed by a factor situated at a point x . When consider this result, it is convenient to take the body as the gas in a rigid and closed surface and the factor as a heat source that efects this gas body. Then C characterizes power of the heat source

Example 1: Let volume of the body be a sphere with center at 0 and radius ρ and let the attracted unit particle be at $P(0,0,z)$, $z>\rho$. Also let be $f(y)=1$ and $\alpha = 1/2$.

In this case the components X, Y of the force due to this gas body and this source will be zero. For the component Z , by using spherical coordinates (a,θ,φ) , we have

$$Z=\int_0^{2\pi}\int_0^{\rho}\int_0^{\pi}\left[\frac{a^2\sin\theta\left(a\cos\theta-z\right)}{\left(a^2+z^2-2az\cos\theta\right)^{3/2}}+\frac{Ca^2\sin\theta\left(a\cos\theta-z\right)}{\left(a^2+z^2-2az\cos\theta\right)^{9/2}}\right]d\theta da d\varphi$$

$$=-\frac{4\pi\rho^3}{3z^2}+C\int_0^{2\pi}\int_0^{\rho}\int_0^{\pi}\left[\frac{a^2\sin\theta\left(a\cos\theta-z\right)}{\left(a^2+z^2-2az\cos\theta\right)^{9/4}}\right]d\theta da d\varphi$$

$$=-\frac{4\pi\rho^3}{3z^2}+\frac{4\pi C}{15z^2}\left[\frac{2z^2}{\left(z+\rho\right)^{3/2}}-\frac{9z}{\left(z+\rho\right)^{1/2}}+\left(z-\rho\right)^{1/2}\left(\frac{3}{\left(z-\rho\right)}+10z+5\rho-3\right)\right.$$

$$\left.+5\left(z+\rho\right)^{3/2}+\left(15z+3\right)\left(z+\rho\right)^{1/2}+10z^{1/2}-30z^{3/2}\right] \tag{2.1}$$

without the heat source the component Z of the force is obtained as follows:

$$Z=-\frac{4\pi\rho^3}{3z^2}=-\frac{M}{z^2} \tag{2.2}$$

this says us that the body is concentrated at its center. With the source, according to Newton's gravitation rule since the magnitude of the force due to a body must be directly proportional to the its mass, by taking parentheses $\frac{4\pi\rho^3}{3}$ of the result in (2.1), we see that the point where body is concentrated is changed by the source. Also if we take the source situated some points $P\left(0,0,z\right)$ to infinity on the $z-$axes, we have the result in (2.2) as expected.

By using similar mind we can also have a interpretation of the Riesz Potential on the velocity field of any fluid. To do this we must give some notions.

III. Velocity Fields and Riesz Potential

Now let's consider velocity field of a fluid in place of Newtonian field. The motion of a single particle may be described by giving its coordinates as function of time:

$$x=x(t)\ \ ,\ \ y=y(t)\ \ ,\ \ z=z(t)$$

If, however, we have a portion of a gas, liquid or elastic solid in motion, we must have such a set of equations or the equivalent, for every particle of the medium. To be more specific, let us talk of a fluid. The particles of the fluid may be characterized by their coordinates at any given instant, say $t=t_0$. Then the equations of all the paths of the particles may be united in a single set of three, dependent on three constants:

$$x=x(x_0,y_0,z_0,t)\ \ ,\ \ y=y(x_0,y_0,z_0,t)\ \ ,\ \ z=z(x_0,y_0,z_0,t) \tag{3.1}$$

for these will tell us at any instant the exact position of the particle of the fluid which at t_0 was at (x_0,y_0,z_0) . The functions occurring in these equations are supposed to satisfy certain requirements as to continuity, and the equations are supposed to be solvable for x_0,y_0 and z_0 . In particular, x must reduce to x_0 , y to y_0 and z to z_0 when $t=t_0$. The velocities of the particles are the vectors whose components are the derivatives of the coordinates with respect to the time :

$$\frac{dx}{dt}=x'(x_0,y_0,z_0,t)\ \ ,\ \ \frac{dy}{dt}=y'(x_0,y_0,z_0,t)\ \ ,\ \ \frac{dz}{dt}=z'(x_0,y_0,z_0,t) \tag{3.2}$$

These equations give the velocity at any instant of a particle of the fluid in terms of its position at $t=t_0$. It is often more desirable to know the velocity at any instant with which the fluid is moving past a given point of space. To answer such a question, it would be necessary to know where the particle was at $t=t_0$ which at the given instant t is passing the given point $\left(x,y,z\right)$. In other words, we should have to solve the equations (3.1) for x_0,y_0,z_0 . The equations (3.2) would then give us the desired information. Let us suppose the steps carried out once for all, that is, the equations (3.1) solved for x_0,y_0,z_0, in terms of x,y,z and t , and the results substituted in (3.2). We obtain a set of equations of the form

$$\frac{dx}{dt}=X(x,y,z,t)\ ,\ \frac{dy}{dt}=Y(x,y,z,t)\ ,\ \frac{dz}{dt}=Z(x,y,z,t)$$

The right hand members of these equations define the velocity field[1]. As the field is changing, there will be one set of field lines at one instant and another at another. We mean by the field lines,a family of curves depending on the time, which at any instant have the direction of the field at every point at that instant. The lines of flow can be found like Newton field lines:

$$\frac{dx}{X(x,y,z,t)}=\frac{dy}{Y(x,y,z,t)}=\frac{dz}{Z(x,y,z,t)} \tag{3.3}$$

on the assumption that t is constant.
Divergence of the velocity field of a flow

$$divV=\nabla V=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}$$

states flux density of the flow. Integral of this over a control volume gives total flux of the flow. If this field is conservative, then we have expression

$$rotV=\nabla\times V=\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x},\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)=0$$

which state that the flow is irrotational. If a irrotational flow occupy a simply connected flow region (It show that the field is conservative), then there exist a scalar field ϕ which is called velocity potential for the flow such that

$$V=\left(X,Y,Z\right)=\nabla\phi\,.$$

Now we give physical interpretation of the field corresponding to the sum of a velocity potential and the Riesz potential $n=3,\ \ 0<\alpha<3$. Also we take Riesz potential as

$$R=C_3^{\alpha}\iiint_V\frac{f(v)}{r^{3+k-\alpha}}dv$$

where k is a suitable constant and $r=|u-v|$. In this case, let's consider function that is linear combination of a potential and the Riesz potential:

$$\Phi=\phi+CR=\phi(u)+C_{\alpha}\iiint_V\frac{f(v)}{r^{3+k-\alpha}}dv\quad ,\quad C_{\alpha}=CC_3^{\alpha}$$

where ϕ is velocity potential of field of a flow that is been irrotational, C is a constant. Corresponding that we have velocity field as

$$\mathbf{V}=\nabla\Phi(u)=\nabla\phi(u)-\iiint_V\left[\tilde{C}_{\alpha}\frac{u-v}{r^{5+k-\alpha}}\right]f(v)dv\ ,\ \ \tilde{C}_{\alpha}=(3+k-\alpha)C_{\alpha}\,.$$

Let us take divergence of this f,ield in order to control flux density of the fluid. Then we obtain

$$\nabla\cdot\mathbf{V}=\Delta\Phi=\Delta\phi+\tilde{\tilde{C}}\iiint_V\left[\frac{2+k-\alpha}{r^{5+k-\alpha}}\right]f(v)dv\ \ ,\ \ \tilde{\tilde{C}}=\tilde{C}_{\alpha}\left(2+k-\alpha\right)$$

where V is volume of source that will be determined later. This shows that flux density of the fluid is changed. Here our fluid and second term which was formed by Riesz potential can be thought as any gas body and as corresponding in the field of a source which effects this fluid such that as source approaches the fluid,the changing at flux is increased and as source diverges the fluid,the changing at flux is decreased. According to this review, it is convenient to think this source as a heat source. In this case V is volume of the heat source. Therefore we can think that the Riesz potential which formed the second term play a role in such system.

Example 2: To illustrate the above considerations,let us examine the flow given by

$$x=x_0+t\ \ ,\ \ y=y_0e^t\quad ,\ \ z=z_0$$

Here x,y,z reduce to x_0,y_0,z_0 for $t=t_0=0$. The velocities of given particles are furnished by

$$\frac{dx}{dt}=1\ ,\ \frac{dy}{dt}=y_0e^t\ \ ,\ \ \frac{dz}{dt}=0$$

and the differential equations of the flow are obtained from these by eliminating x,y and z

$$\frac{dx}{dt}=1\ ,\ \frac{dy}{dt}=y\ ,\ \frac{dz}{dt}=0$$

The field $\mathbf{V}=(X,Y,Z)=(1,y,0)$ is stationary, since the velocities at given points are independent of the time. Now let us take Riesz potential that represents a heat source as

$$C_3^{1/2}\iiint_V \frac{1}{r^{5/2+k}}dv$$

with $\alpha=1/2$ and $f(v)=1$ in equation. The divergence of this flow field is found as

$$\nabla\cdot\mathbf{V}=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=1\,.$$

If the Riesz potential is included into the system, then the flow field is

$$\mathbf{V}=\nabla\Phi(u)=\nabla\phi(u)-\iiint_V\left[\tilde{C}_\alpha\frac{u-v}{r^{5+k-\alpha}}\right]f(v)dv=(1,y,0)-\iiint_V\left[\tilde{C}\frac{u-v}{r^{9/2+k}}\right]dv$$

and flux of flow is

$$\nabla\cdot\mathbf{V}=\Delta\Phi=\Delta\phi-\tilde{\tilde{C}}\iiint_V\left[\frac{2+k-\alpha}{r^{5+k-\alpha}}\right]f(v)dv=1-\tilde{C}\iiint_V\frac{dv}{r^{9/2+k}}\,.$$

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