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From Pell to Pell$^+$ and CDS Sequences and Salient Features of Their Generator Matrices

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Abstract: *This research paper, in the first part, introduces two new sequences- 'CDS' sequence and Pell$^+$sequence. These sequences, in order, introduced are identified and derived as a result of identifying recurrence relation in fundamental geometric properties of right triangles of Fermat family and as an attempt of locating such Pythagorean triplets whose middle term itself is a square of an integer. In addition to this, some algebraic properties related to these sequences are also established. These sequences have established their importance when we extended our work on operator matrices on Fermat family triplets of Right triangles. Immediate application of the operator matrices and their algebraic properties, like Eigen values of different exponents will be best summarized by the terms of these new sequences.*
Keywords: *CDS Sequence, Fermat Family, Generator Matrices, Pell Sequence, Pell $^+$ Sequence*

I. Introduction

In this article we initially introduce Fermat family of right triangles and mention its salient features. The next part of the work introduces three sequences- Pell sequence, Pell$^+$ sequence, and CDS sequence. Along with introduction of each sequence, its salient features are highlighted. Pell sequence has a strong historical back ground and its soundness lies in the enormous literature mathematicians have followed since then till date. The remaining two sequences which are attempted as an output or outcome from Pell do stand on their own merits and claim fairly equal share on Pell sequence. This proves that derivation from Pell is just a consequence but its independent existence can never be side -lined. It has always remained a basic pattern to give formulae for both the nth term and recurrence relation of a given sequence under discussion and dominantly these two features provide the sound metal base for further details. The next important and useful role that concerns is regarded in the study of generator matrices and their algebraic properties. Characteristic roots of generator matrices of different but successive exponentiations are again closely associated with the basic sequences in many cases or can be partly inter-related with that of terms of some known sequence; which is reflected in case of CDS and Pell$^+$ sequence in the note that follow.
We study all this in order beginning with right triangles of Fermat family.

1.1 Fermat Family of Right Triangles

A right triangle R_i $where \quad i = 1,2,3, \ldots$ is said to be the i^{th} member of Fermat family if for some $a_i, b_i \& h_i \in N$ with the following conditions satisfied

1. $a_i < b_i < h_i$
2. $(a_i,\ b_i) = 1$
3. $b_i - a_i = 1$
4. $a_i^{\ 2} + b_i^{\ 2} = h_i^{\ 2} \quad \forall \text{ i} \in N$ …… (1)

We shall call a_i – shorter leg of Right triangle R_i
We denote the infinite set of all such right triangles of Fermat family by F.
$F = \{R_i | R_i = R_i(a_i, b_i, h_i) \quad \forall i \in N\}$
Each R_i satisfies all four above mentioned properties. We identify some members of the set F

Table-1

R_i Triangle Number	a_i Shorter Leg	b_i Next Leg	h_i Hypotenuse
R_1	$a_1 = 3$	$b_1 = 4$	$h_1 = 5$
R_2	$a_2 = 20$	$b_2 = 21$	$h_2 = 29$

R_3	$a_3 = 119$	$b_3 = 120$	$h_3 = 169$
R_4	$a_4 = 696$	$b_4 = 697$	$h_4 = 985$

There are many interesting properties associated with the corresponding sides of these triangles. One can visualize these sides a_i, b_i, and h_i as an infinite and strictly monotonically increasing sequences following certain mathematical pattern.

1.2 Three Sequences

In this section, we will discuss three different sequences which are closely connected, in a way or other, with some geometric properties of the above mentioned right triangles of Fermat family.

1.2.1 Pell Sequence

Pell numbers arise historically and most notably appear in the infinite sequence that converge to the better approximation to the irrational number $\sqrt{2}$. If two positive integer $x \,\&\, y(x, y \to \infty)$ which are alternatively the solution of $x^2 - 2y^2 = \mp 1.$ Their ratio $\frac{x}{y}$ is claimed to provide better and better solution to where $\sqrt{2}$ is on real line.

The sequence is $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \dots$ …… (2)

In this sequence the denominator of each fraction progressively is a Pell number and the sequence is a Pell sequence.

i.e. 0, 1, 2, 5, 12, 29, 70 … is a Pell sequence.

From this, we write the recurrence relation as follows:

$$P_{n+2} = 2P_{n+1} + P_n \ \ for\ n \geq 0\ with\ P_0 = 0, P_1 = 1$$

Using the same recurrence relation we derive its general term:

$$P_n = P(n) = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right] \ \ for\ \ n \in N \cup \{0\}$$

It is also obvious to claim that the sequence of difference $(x - y)$ for each term in the above sequence of rations is also a Pell sequence.

This allows inclusion of zero and which symmetries the formation of the Pell sequence which we are going to use.

Form (2) the sequence of $(x - y)$ is 0, 1, 2, 5, 12, 29 …

We write it as:

$$P_n = \begin{Bmatrix} 0\ if\ n = 0 \\ 1\ if\ n = 1 \\ 2P_{n-1} + P_{n-2}\ if\ n \geq 2 \end{Bmatrix}$$

It can also be visualized as a sequence $\left(\frac{x_i}{y_i}\right)\ \ for\ all\ i \in N$

Where $x_1 = 1\ , x_2 = 3\ and\ x_n = 2x_{n-1} + x_{n-2}\ for\ n \geq 3$

And $y_1 = 1\ , y_2 = 2\ and\ y_n = 2y_{n-1} + y_{n-2}\ for\ n \geq 3$

It gives relatively a better mathematical felling that,

$$\lim_{n \to \infty}\left(\frac{x_n}{y_n}\right) = \sqrt{2}$$

Where the terms of the sequence (x_n) and (y_n) observes the same recurrence relation.

1.2.2 Pell + Sequence

In our work on Pythagorean triplets, right triangles of Fermat family, and on performing algebraic operations on generator matrices of different sequences, we have observed and identified some special type of recurrence which is closely related to the consecutive terms of Pell sequence. In the next section of this paper working on operator matrices and in the previous work on infinite sequences and infinite matrices (not in this article) we have used terms of the sequence, we mention here as Pell+ sequence.

It is a sequence, denoted as $P^+(n)\ or\ P_n^+$, whose terms are the sum result of distinct pair of consecutive terms of Pell sequence:

i.e. $P^+(n) = P_n^+ = P_{n-1} + P_n \ \ ,\ n \in N$[where P_0 =0] …… (3)

Where P_n^+ and P_n are the n^{th} term of Pell+ sequence and Pell sequence respectively.

This helps write some terms of Pell + Sequence. Some terms of P_n^+ sequence are

1, 3, 7, 17, 41, 99… … (4)

It is interesting to note that same recurrence relation is continued to satisfy the terms of the Pell+ sequence:

i.e. $P_{n+2}^+ = 2P_{n+1}^+ + P_n^+ \ ,\ \forall\, n \in N\ with\ \ P_1^+ = 1, P_2^+ = 3$

Using this recurrence relation we define its general term of P_n^+ sequence:

$$P_n^+ = P^+(n) = \frac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]\ \ for\ \ n \in N \qquad \text{…… (5)}$$

Salient Features of P_n^+ Sequence:

There are many interesting results associated with P_n^+ sequence. We mention some salient features of P_n^+ sequence.

*1 Generator Matrix

It is known that generator matrix to a given sequence helps identifying many inherent properties of the sequence; we look for the sequence on hand.

Considering the recurrence relation in the terms of the sequence the generator matrix can devised as $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Using this generator matrix along with the first two terms $\begin{bmatrix} P_2^+ \\ P_1^+ \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = B$

$$A * B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} P_3^+ \\ P_2^+ \end{bmatrix}$$

$$A^2 * B = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \end{bmatrix} = \begin{bmatrix} P_4^+ \\ P_3^+ \end{bmatrix}$$

$$A^3 * B = \begin{bmatrix} 12 & 5 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 41 \\ 17 \end{bmatrix} = \begin{bmatrix} P_5^+ \\ P_4^+ \end{bmatrix}$$

And continuing in the same way, without loss of generality we write;

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} P_2^+ \\ P_1^+ \end{bmatrix} = \begin{bmatrix} P_{n+2}^+ \\ P_{n+1}^+ \end{bmatrix}$$

*2 Eigen values of the generator matrix

In this section we find Eigen values of generator matrices and its exponent matrices. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ be the generator matrix of the Pell + sequence. We list Eigen values as follows.

For matrix A $= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$; Eigen values $= 1 \pm 1\sqrt{2}$

For matrix $A^2 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$; Eigen values $= 3 \pm 2\sqrt{2}$

For Matrix $A^3 = \begin{bmatrix} 12 & 5 \\ 5 & 2 \end{bmatrix}$; Eigen values $= 7 \pm 5\sqrt{2}$

At this point, before we lay down eigen values for A^n, we identify the pattern in the flow of both the parts of each eigen value; the first one in the flow is the corresponding term of the Pell+ sequence and the one in the second part is the corresponding term of Pell sequence.

In general for $A^n = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n$ Eigen values $= \boldsymbol{P_n^+ \pm P_n\sqrt{2}}$ …… (6)

There are many interesting mathematical properties associated with sequence and we will mention them in the sections to follow.

1.2.3 CDS Sequence

This is the sequence observed and identified by a group of research students working on properties of Pythagorean triplets. It is a sequence of some natural numbers in the Pythagorean primitive triplet (a, b, h) where the number 'a' – the shorter leg in a right triangle is such that $a^2 + b^2 = h^2$ and $\mathbf{b} = \boldsymbol{x^2}$ for some $x \in N$ i.e. $a^2 + x^4 = h^2$

The following TABLE-2 clarifies the notion

Table-2

Sr. No.	Triangle with Sides		
	a	b= x^2	h
1	3	4 =$(2)^2$	5
2	17	144=$(12)^2$	145
3	99	4900=$(70)^2$	4901
4	577	166464=$(408)^2$	166465

Some terms of CDS sequence are 3, 17, 99, 577, 3363… …… (7)

Another approach to view at the terms of the sequence is that the terms are even ordered terms of Pell+ sequence:

i.e. $CDS(n) = P^+(2n) = P(2n) + P(2n-1) \quad \forall n \in N$ …… (8)

The recurrence relation in the terms of the sequence is:

$t_{n+2} = 6t_{n+1} - t_n \quad for\ n \in N\ ,\ with\ t_1 = 3\ , t_2 = 17$ …… (9)

Using this recurrence relation we derive its general term:

$CDS(n) = \frac{1}{2}\left[\left(3 + 2\sqrt{2}\right)^n + \left(3 - 2\sqrt{2}\right)^n\right]\ for\ n \in N$ …… (10)

The terms of this sequence are observed in the Eigen values of the generator matrices of many sequences and also in the operator matrices of many mathematical systems. The next section is devoted on the same line.

II. The First Operator Matrix

At this stage, we recall the section 1.1 of the introduction to primitive triplets of the Fermat family. We briefly state that for the positive integers $a, b\ \&\ h$ if

1. $a < b < h$
2. $(a, b) = 1$
3. $b - a = 1$ and
4. $a^2 + b^2 = h^2$ …… (11)

conditions hold then (a, b, h) is a Fermat triplet. The content of the TABLE-1 shows right triangles R_1, R_2, R_3 ... of Fermat family.

This inspires to work on

1. Construction of $3 \times n$ matrix of Fermat triplet.
2. To search for an operator matrix which on multiplication with the matrix of Fermat triplets yields the next matrix of the Fermat triplets whose columns entries have advanced by the immediate next triplet of the next Fermat triangle R_{n+1}.

Handling the first point, we have the Fermat matrix of the type

$FM_{13} = [R_1(a_1,b_1,h_1)R_2(a_2,b_2,h_2)R_3(a_3,b_3,h_3)]$

$i.e. FM_{13} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ h_1 & h_2 & h_3 \end{bmatrix}$ …… (12)

Where all $a_i\ , b_i\ , h_i\ (i = 1, 2, 3)$ observe the protocol shown by the relation (11)

In the same way, we have $FM_{14} = [R_1R_2R_3R_4]$, where $R_i = R_i(a_i, b_i, c_i)$ observes the same protocol (11).

2.1 The Fermat Operator

The Fermat operator matrix of the first type is denoted as $JM1$, is a non-singular matrix.

i.e. $|JM1| \neq 0$. The dominating property of $JM1$ is that when it pre-multiplies the Fermat matrix the resultant is also a Fermat matrix but it shifts the column entries of triangle by the next triangle in the sequence.

i.e.$(JM1)\big(FM_{13}(R_1R_2R_3)\big) = FM_{23}(R_2R_3R_4)$ and

$$(JM1)\big(FM_{14}(R_1R_2R_3R_4)\big) = FM_{24}(R_2R_3R_4R_5)$$

We generalize the above mentioned result as follows.

$$(JM1)\big(FM_{1n}(R_1R_2 \ \dots \ R_n)\big) = FM_{2n}(R_2R_3 \ \dots \ R_{n+1}) \qquad \dots\dots (13)$$

This non-singular Fermat operator matrix is

$$JM1 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

We illustrate this by an example.

Let $FM_{13} = [R_1R_2R_3] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ h_1 & h_2 & h_3 \end{bmatrix} = \begin{bmatrix} 3 & 20 & 119 \\ 4 & 21 & 120 \\ 5 & 29 & 169 \end{bmatrix}$

The product $(JM1)(FM_{13}) = \begin{bmatrix} 20 & 119 & 696 \\ 21 & 120 & 697 \\ 29 & 169 & 985 \end{bmatrix} = FM_{23}$

i.e. $(JM1)\big(FM_{13}(R_1R_2R_3)\big) = FM_{23}(R_2R_3R_4)$

In the same way we can derive the next results by using algebraic properties of matrix algebra.

$$(JM1)^2\big(FM_{13}(R_1R_2R_3)\big) = FM_{33}(R_3R_4R_5)$$

$(JM1)^3 FM_{13}(R_1R_2R_3) = FM_{43}(R_4R_5R_6)$ Etc.

It also works on Fermat matrices of order 3xn.

In general we write that

$$(JM1)^2\big(FM_{1n}(R_1R_2R_3 \ \dots . R_n)\big) = FM_{3n}(R_3R_4R_{5 \dots\dots\dots} R_{n+2}) \qquad \dots\dots (14)$$

This property can be extended.

2.2 Properties of $JM1$ Matrix

In above section, we have seen the important factor of the non-singular matrix $JM1$

$$JM1 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

Some more algebraic properties of the above matrix are noted below.

*1 $Det. A = |JM1| = 1$

*2 In this property we are interested to show that how the different powers of the matrix $JM1$ are closely related to the Pell[+] sequence.

From $JM1 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ we have $(JM1)^2 = \begin{bmatrix} 9 & 8 & 12 \\ 8 & 9 & 12 \\ 12 & 12 & 17 \end{bmatrix}$

We generalize the result of exponentiation and coordinate the entries with corresponding terms of Pell[+] and Pell sequence.

$$(JM1)^n = \begin{bmatrix} \frac{P_{2n}^{+}+1}{2} & \frac{P_{2n}^{+}-1}{2} & P_{2n} \\ \frac{P_{2n}^{+}-1}{2} & \frac{P_{2n}^{+}+1}{2} & P_{2n} \\ P_{2n} & P_{2n} & P_{2n}^{+} \end{bmatrix} \quad \ldots\ldots \ (15)$$

Where P_n^{+} shows the Pell^{+} sequence.

*3 It is known that the Eigen values and related properties are very closely associated with exponentiation of the given root matrix $JM1$. We mention the same point here but with a different view point. We observe and state that the Eigen values are closely connected with two sequences which we have mentioned above – Pell sequence and Pell^{+} sequence.

Table-3

Matrix	Eigen Values		
$JM1$	$3-2\sqrt{2}$	1	$3+2\sqrt{2}$
$(JM1)^2$	$17-12\sqrt{2}$	1	$17+12\sqrt{2}$
$(JM1)^3$	$99-70\sqrt{2}$	1	$99+70\sqrt{2}$

The observation and its related mathematics simplify the above notion.
We generalize that,
If e_{1n}, e_{2n}, e_{3n} are three consecutive Eigen values in ascending order of $(JM1)^n \ \forall n \in N$ then

1. $e_{1n} = P_{2n}^{+} - (P_{2n})(\sqrt{2})$
2. $e_{2n} = 1$
3. $e_{3n} = P_{2n}^{+} + (P_{2n})(\sqrt{2})$

*4 At this stage it is a noticeable point to state that the inverse of the Fermat operator matrix $JM1$, denoted as $(JM1)^{-1}$, and its different positive integral positive exponents, i.e. $(JM1)^{-n}$ exhibit just the opposite properties to that of $(JM1)^n$.

We, without going in to more details, state the inverse of the primary Fermat operator $JM1$.

$$(JM1)^{-1} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

As shown above in the case of generalization, i.e. $(JM1)^n$ in terms of known sequences, we can also express $(JM1)^{-n}$ in terms of the same known sequences.

III. Conclusion

Echoes of partial soundness in the mathematical discussion in the above sections allegedly claim some mathematical pursuant that barring certain sequences most of the sequences are or can be established to follow some inter-relationship. Their soundness reflects in generator matrices which are indirectly associated with recurrence relation within their positional values. In addition to this, inter-woven relationship amongst their Eigen values binds the structural relation between them. Member triangles of Fermat family and their trigonometrical properties format sound algebraic insight.

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