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Application of Rao Transform to find the Numerical Solution of Integral Equations with Linear Kernel

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Abstract: *This paper presents an application of the novel modern Rao technique in attempt to solve integral equations of the second kind with variable limits of integration (i.e Volterra Equation of the second kind) when the kernel being linear in the arguments x and t. The numerical solution obtained by applying such technique has been compared with the exact solution that is obtained analytically. An algorithm for Rao-transform is contained here with an illustrative example, which has been solved by using MATLAB, is presented at the end of this paper.*

Keywords*: Volterra integral equations (VIE), linear kernel, Rao technique and Rao-Transform, ordinary differential equations.*

I. Introduction

The exact solution of VIE of the second kind can be deduced analytically in various methods cf.[1]. Such equations usually arise in many branches of applicable sciences such as electromagnetic fields and fluids and dynamics…etc. Hereby we intend to solve the same sort of VIE with linear kernel numerically by applying Rao-transform technique [2].

Consider the following VIE of the second kind:

$$g(x)= f(x)+\lambda\int_a^x k(x,t)f(t)dt \tag{1}$$

where all the functions appear in equation (1) are assumed to be continuous, integrable and differentiable up to some order. The kernel of this integral equation is considered to be linear in the arguments *x* and *t*.

II. Abbreviation of Rao Technique [2]

The general linear integral equation [1,2,3] including VIE of the second kind is:

$$g(x)= f(x)+\int_r^s k(x,t)f(t)dt \tag{2}$$

where the disappearance of λ before the integral is considered to be $\lambda=1$ without loss of generality and all the functions appeared in this equation preserve the same properties of those in equation (1).

Set $u = x-t$, then equation (2) becomes:

$$g(x)= f(x)+\int_{x-s}^{x-r} k(x,u)f(u)du \tag{3}$$

which is called Rao-transform.

Following the procedure below, equation (3) can be converted to a differential equation:

$$g(x)= f(x)+\int_{x-s}^{x-r} k(x,x-t)f(x-t)dt$$

Expand the function $f(x-t)$ by Taylor-series and rearrange the terms as:

$$g(x) \approx f(x) + \sum_{n=0}^{N}\frac{d^n(x)}{dx^n}\cdot\frac{(-1)^n}{n!}\int_{x-s}^{x-r} t^n k(x,x-t)dt$$

the last equation can be written as:

$$g(x) \approx f(x) + \sum_{n=0}^{N} k_n(x)f^{(n)}(x) \tag{4}$$

where $k_n(x)=\dfrac{(-1)^n}{n!}\int_{x-s}^{x-r} t^n k(x,x-t)dt$

Let $f^{(m)}(x)=0$ when $m>N$.

For m^{th} order derivative of $g(x)$, $m=0,1,\ldots,N$, equation (4) implies:

$$g(x) = f(x) + \sum_{n=0}^{N}\sum_{p=0}^{m} k_n^{(m-p)}(x) f^{(n+p)}(x)$$

simplify and rewrite the last equation as:

$$g^{(m)}(x) = f^{(m)}(x) + \sum_{n=0}^{N} k_{m,n}(x) f^{(n)}(x),\ \text{for } m=0,1,\ldots,N \qquad (5)$$

where

$$k_{m,n}(x)= \sum_{n=0}^{min(n,m)} C_p^m k_{n-p}^{(m-p)}(x) \qquad (6)$$

equation (5), in virtue of equation (6), can be written as a compact vector-matrix form:

$$\boldsymbol{g_x} = (\boldsymbol{I}+\boldsymbol{K_x})\boldsymbol{f_x} = \boldsymbol{J_x f_x} \qquad (7)$$

where $\boldsymbol{g_x}$ is the column vector of the derivatives $g^{(m)}$, $m=0,1,\ldots,N$.

$\boldsymbol{I}$ is the $\boldsymbol{N}\times\boldsymbol{N}$ identity matrix,

$\boldsymbol{K_x}$ is the $\boldsymbol{N}\times\boldsymbol{N}$ matrix deduced from equation (6), and

$\boldsymbol{f_x}$ is the column matrix for the derivatives of the function f, $m=0,1,\ldots,N$.

Multiplying equation (7) by $\boldsymbol{J_x}^{-1}$ from the left to get:

$$\boldsymbol{f_x} = \boldsymbol{J_x}^{-1}\boldsymbol{g_x} \qquad (8)$$

where $\boldsymbol{J_x}^{-1}=(\boldsymbol{I}+\boldsymbol{K_x})^{-1}$ will be denoted by $\boldsymbol{K_x}^*$.

equation (8) introduces the sought for solution explicitly as:

$$f^{(0)} = f(x) \approx \sum_{n=0}^{N} k_{0n}^{*}(x) g^{(n)}(x) \qquad (9)$$

where k_{0n}^* are the n entries of the matrix $\boldsymbol{K_x}^*$.

This equation gives the approximate numerical solution for VIE (2).

Creative Algorithm

The algorithm for Rao technique is given below and the equivalent MATLAB6.5 program is created as a package to solve all of the integral equations of the same class.

Rao Algorithm

INPUT : r,s = Interval $[r,s]$.

N = Dimension of Matrix $\boldsymbol{K_x}$.

$k(x,t)$.

$g(x)$.

PROCESS : $h = (s-r)/N$

$$k_n(x)=\frac{(-1)^n}{n!}\int_0^x t^n k(x,x-t)dt$$

$$\boldsymbol{K_x} = \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{p=0}^{min(m,n)} C_p^m \frac{d^{(n-p)}}{dx^{(n-p)}} K_{m-p}(x)$$

$\boldsymbol{J_x} = \boldsymbol{I}+\boldsymbol{K_x}$

$\boldsymbol{K_x}' = \boldsymbol{J_x}^{-1} = \text{inv}(\boldsymbol{J_x}(i))$

$$f_x^{(0)}(x_i) = \sum_{n=0}^{N} k_{0n}' g^{(n)}(x_i)$$

OUTPUT : **FOR** $i=0:N$

$x_i = a+ih$

error = **abs**($f(x_i)-f_0(x_i)$)

PRINT (x_i, $f(x_i)$, $f_0(x_i)$, error)

ENDFOR $\{i\}$

END.

4. Illustrative Example

Hereby, we submit an example with it's necessary explanation to show how does the Rao-transform and his technique can be applied and in addition, to show the efficiency of our program and its obtained accuracy.

We shall demonstrate the analytic solution, firstly, for the sake of comparison with the numerical one.

4.1 Analytical Solution of VIE of the Second Kind

Consider the following Volterra integral equation of the second kind [5]:

$$e^{x}-3x-\frac{1}{2}=f(x)-\int_{0}^{x}(5-6x+6t)f(t)dt \tag{10}$$

Differentiate twice analytically equation (10) to convert it into the O.D.E.:

$$f''(x)-5f'(x)+6f(x)=e^{x} \tag{11}$$

then using appropriate analytic method to solve equation (11) by writing the equivalent characteristic algebraic equation:

$$r^2-5r+6=0 \tag{12}$$

finding the roots of equation (12) as:

$(r-2)(r-3) = 0$,

which implies that the roots are:

$r = 2$ or $r = 3$

hence, the complementary solution will be:

$$f_c = c_1e^{2x} + c_2e^{3x} \tag{13}$$

To obtain the particular solution,

Let $f_p(x) = Ae^{x}$, (14)

then equations (12) and (14) yields:

$A - 5A + 6A = 1$,

from which we get:

$A=\frac{1}{2}$

Substituting this value of A in equation (14) produces the particular solution in the form:

$$f_p(x)=\frac{1}{2}e^{x} \tag{15}$$

then the general solution f_g of equation (11) is obtained by the fact that $f_g = f_p + f_c$ i.e. by summing up equations (13) and (15) to get:

$$f_g=\frac{1}{2}e^{x}+c_1e^{2x}+c_2e^{3x}$$

at $x = 0$, the value of $f_g(0) = \frac{1}{2}$, and $f'(0) = \frac{1}{2}$ whence, $c_1=c_2=0$, therefore, $f_g = \frac{1}{2}e^{x}$.

4.2 Numerical Solution of VIE

Consider the same (VIE) that has been given in subsection (4.1) i.e:

$$e^{x}-3x-\frac{1}{2}=f(x)-\int_{0}^{x}(5-6x+6t)f(t)dt \tag{16}$$

Rewrite this equation as:

$$e^{x}-3x-\frac{1}{2}=f(x)+\int_{0}^{x}(6x-5-6t)f(t)dt \tag{17}$$

where the linear kernel $k(x,t) = 6x-5-6t$

Using the proper substitution $u=x-t$ to get:

$k(x,x-t) = 6x-5-6(x-t) = 6t-5$

so equation (17) becomes:

$$e^{x}-3x-\frac{1}{2}=f(x)+\int_{0}^{x}(6t-5)f(x-t)dt$$

Appling Rao-transform to obtain:

$$k_n(x) \equiv \frac{(-1)^n}{n!}\int_{x-1}^{x} t^n k(x, x-t)dt, \text{ for } n=0,1,2.$$

which implies that:

$k_0(x) = 3x^2-5x$,

$$k_1(x) = \frac{5}{2}x^2 - 2x^3,$$

$$k_2(x) = \frac{3}{4}x^4 - \frac{5}{6}x^3.$$

Denote by:

$$k_n^{(m-p)}(x) = \frac{d^{(m-p)}}{dx^{(m-p)}}k_n(x), \text{ for } m=0,1,2.$$

From which calculate:

$$k_{m,n}(x) = \sum_{p=0}^{min(n,m)} C_p^m k_{n-p}^{(m-p)}(x) \tag{18}$$

where $C_p^m = \dfrac{m!}{p!(m-p)!}$

Formula (18) gives the following set of functional values:

$k_{00}(x) = 3x^2-5x$,

$$k_{01}(x) = \frac{5}{2}x^2 - 2x^3,$$

$$k_{02}(x) = \frac{3}{4}x^4 - \frac{5}{6}x^3.$$

and

$k_{10}(x) = 6x-5$,
$k_{11}(x) = 3x^2$,
$k_{12}(x) = x^3$, and
$k_{20}(x) = 6$,
$k_{21}(x) = -5$,
$k_{22}(x) = 0$.

Arrange all the above functional values in a matrix form as follows:

$$\boldsymbol{K_x}(x)=\begin{bmatrix} 3x^2-5x & \frac{5}{2}x^2-2x^3 & \frac{3}{4}x^4-\frac{5}{6}x^3 \\ 6x-5 & -3x^2 & x^3 \\ 6 & -5 & 0 \end{bmatrix}$$

On the other hand, for the function g we have:

$$g(x) = e^x - 3x - \frac{1}{2},\ g'(x)=e^x-3 \text{ and } g''(x)=e^x.$$

For which at x=0, 0.5 and 1 we calculate:

$g(0) = 0.5$, $g(0.5) = -0.3513$ and $g(1) = -0.7817$.
$g'(0) = -2$, $g'(0.5) = -1.3513$ and $g'(1) = -0.2817$.
$g''(0) = 1$, $g''(0.5) = 1.6487$ and $g''(1) = 2.7183$.

Up to this end we could solve the compact matrix form:

$\boldsymbol{f_x} = \boldsymbol{J_x}^{-1}\boldsymbol{g_x}$, where $\boldsymbol{J_x} = (c\boldsymbol{I} + \boldsymbol{K_x})$ (non-singular matrix) from which we deduce the approximate solution by virtue of the relation that was given in (9), i.e:

$$f^{(0)} = f(x) \approx \sum_{n=0}^{N} k_{0n}^* g^{(n)}(x)$$

where k_{0n}^* are the entries of the kernel matrix $\boldsymbol{J_x}^{-1}=(\boldsymbol{I}+\boldsymbol{K_x})^{-1} = \boldsymbol{K_x}^*$.

The final approximate result will be:

$f(x) \approx (-0.9473)(-0.7817) + (0.1052)(0.2817) + 0.0263)(2.7183)$
$f(x) = 0.8416$ for $x = 0, 0.5$ and 1 (19)
The exact solution is given by:
$f(0) = 0.5, f(0.5) = 0.8244$ and $f(1) = 1.3592$ (20)
hence the error E_R is obtained as the absolute difference between the numerical and exact values at the mid point x=0.5 from equation (19) and (20).
$E_R = |\, 0.8244\text{-}0.8416 \,| = 0.0172.$

Note: All the calculations below in tables (1,2 and 3) are given up to 8 digits after decimal.

Table 1: The error results depend upon x_i for N=3.

x_i	Exact	Approximate	Error
0.0000	0.50000000	0.50000000	0.00000000
0.3333	0.69780621	0.69741922	0.00038700
0.6667	0.97386702	0.96422838	0.00963864
1.0000	1.35914091	1.33985393	0.01928699

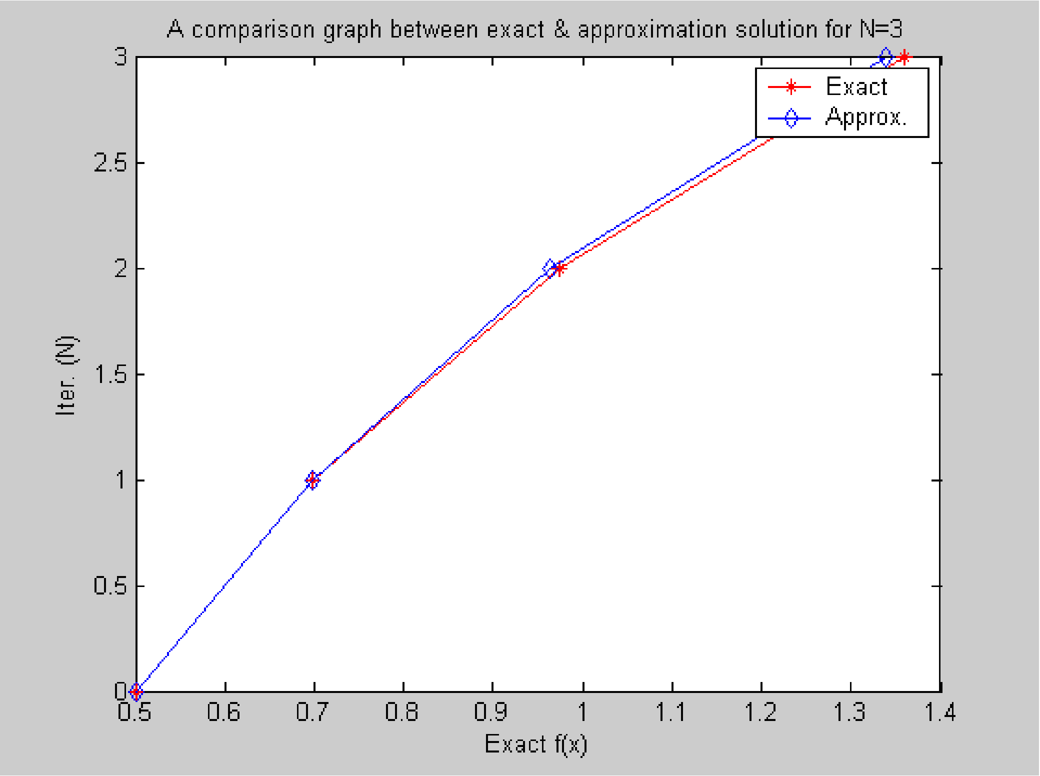

Figure 1: A comparison graph between exact & approximate solution for N=3.

Table 2: The error results depend upon x_i for N=6.

x_i	Exact	Approximate	Error
0.0000	0.50000000	0.50000000	0.00000000
0.1667	0.59068021	0.59068021	0.00000000
0.3333	0.69780621	0.69780626	0.00000004
0.5000	0.82436064	0.82436309	0.00000245
0.6667	0.97386702	0.97392534	0.00005832
0.8333	1.15048795	1.14812198	0.00236596
1.0000	1.35914091	1.35844201	0.00069890

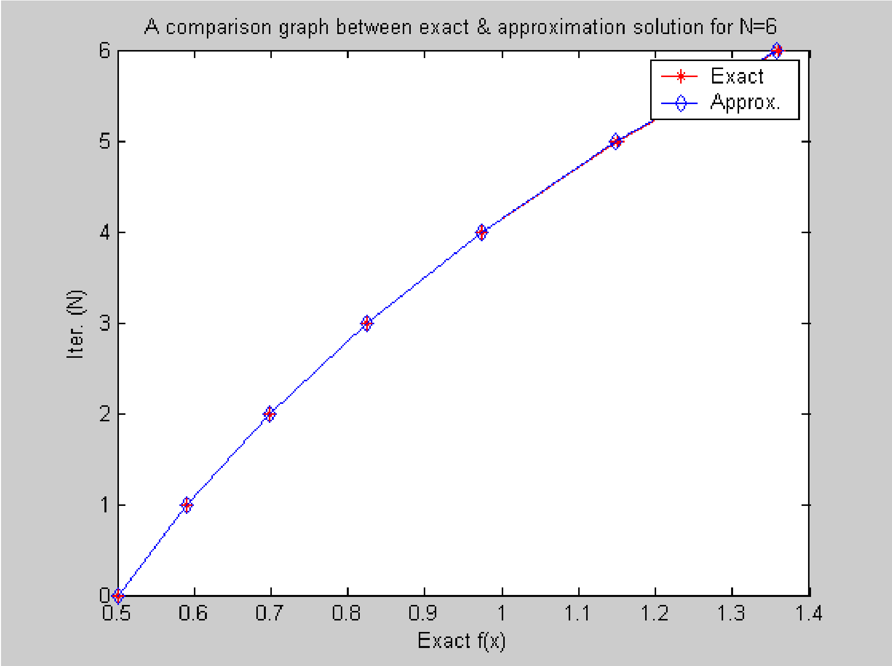

Figure 2: A comparison graph between exact & approximate solution for N=6.

Table 3: The error results depend upon x_i for N=10.

x_i	Exact	Approximate	Error
0.0000	0.50000000	0.50000000	0.00000000
0.1000	0.55258546	0.55258546	0.00000000
0.2000	0.61070138	0.61070138	0.00000000
0.3000	0.67492940	0.67492940	0.00000000
0.4000	0.74591235	0.74591235	0.00000000
0.5000	0.82436064	0.82436064	0.00000000
0.6000	0.91105940	0.91105940	0.00000000
0.7000	1.00687635	1.00687636	0.00000000
0.8000	1.11277046	1.11277048	0.00000001
0.9000	1.22980156	1.22980164	0.00000008
1.0000	1.35914091	1.35914140	0.00000048

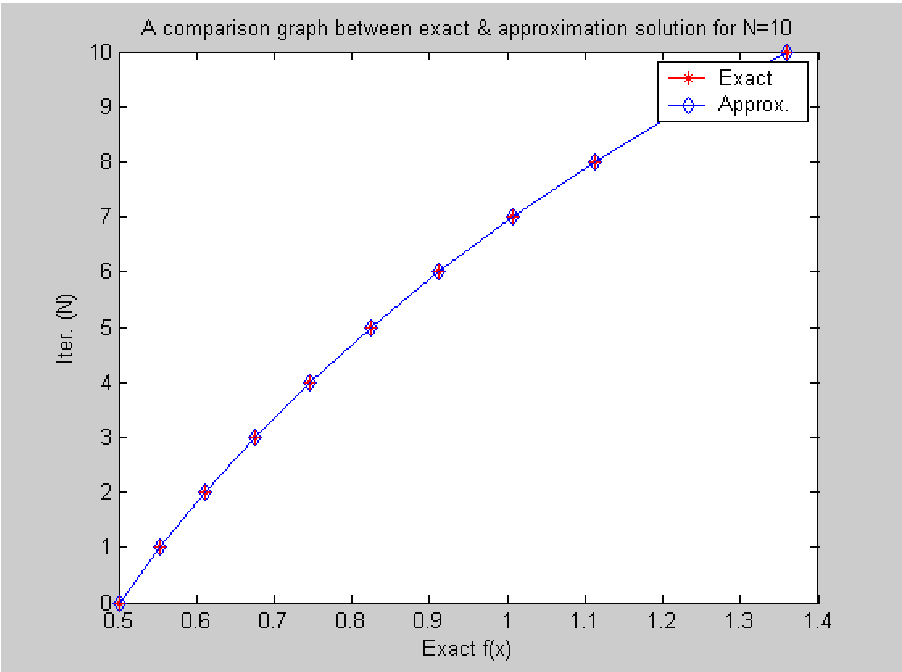

Figure 3:A comparison graph between exact & approximate solution for N=10.

III. Conclusions

From the practical example for the suggested algorithm we conclude the following aspects:

1. From table 1, we notice a significant difference between the exact and approximate solutions (see figure 1).
2. From table 2, we notice a very small difference between the exact and approximate solutions (see figure 2).
3. From table 3, we notice no difference between the exact and the approximate solutions (see figure 3).
4. A comparison study between all the above tables we deduce that when N=10 in table 3 the best result occurs from row0 to row7.

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