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Numerical Solution of nonlinear Singular Systems Using He's Variational Iteration Method

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Abstract: *In this paper, He's Variational Iteration Method (HVIM) is used to study the nonlinear time-invariant and time-varying singular systems. The results obtained using He's Variational Iteration Method and the methods taken from the literature [18] were compared with the exact solutions of the nonlinear time-invariant and time-varying singular systems. It is found that the solution obtained using the He's Variational Iteration Method is closer to the exact solutions of the nonlinear time-invariant and time-varying singular systems. Error Calculations for discrete and exact solutions are presented in a table form to highlight the efficiency of this method.*
Keywords: *Singular systems, nonlinear singular time-invariant, time-varying, He's Variational Iteration Method, Leapfrog Method.*

I. Introduction

Most realistic non-linear systems, in particular singular non-linear systems, do not admit any analytical solution and hence it must be solved using a numerical technique. Conventional methods, such as Euler, Taylor series and Adams-Moultan methods, are restricted to a very small step size in order to obtain a stable solution, which naturally require much computer time. Many new methods have been developed to overcome this step-size constraint imposed by numerical stability, and these are reviewed by Butcher [1] and Murugesan *et al*. [19].

Initially most of the numerical work on non-linear singular systems assumed that the system was an index one. However, many of the problems in constrained mechanics are initially formulated as index one. Singular systems of index up to six naturally occur in mechanics if actuator dynamics, joint flexibility, and other effects are included [2, 3]. Index one non-linear singular systems occur in several other areas [5, 6].

Many numerical methods for non-linear singular systems require that the systems have special structure, such as being a mechanical system with holonomic constraints [15, 20] or have indices of only one or two. A more general approach is introduced in [4]. That approach is explicit and does not preserve constraints. The first approach for the constraint preserving numerical integration of general unstructured higher index singular systems is introduced in [2]. An alternative approach could probably be developed based on the ideas in [7].

In this paper we developed numerical methods for addressing nonlinear singular systems of time-invariant and time-varying cases by an application of the He's Variational Iteration Method which was studied by Sekar and team of his researchers. Recently, Sekar and Nalini [18] discussed the nonlinear singular systems of time-invariant and time-varying cases using Adomian Decomposition Method (ADM). In this paper, the same nonlinear singular systems of time-invariant and time-varying cases problem was considered (discussed by Sekar and Nalini [18]) but present a different approach using the He's Variational Iteration Method with more accuracy for nonlinear singular systems of time-invariant and time-varying cases.

II. Nonlinear Singular Systems

The time-invariant non-linear singular systems of the form is considered

$$K\dot{x}(t) = Ax(t) + f(x(t)) \tag{1}$$

with $x(0) = x_0$, where K is an $n \times n$ singular matrix, A is an $n \times n$ matrix, $x(t)$ is an n - state vector and f is an "n" vector function. In order to make the above system (1) as time-varying case some of the components (not necessarily all the elements) in the system (1) are converted, as time-varying and then the system will be of the following form

$$K(t)\dot{x}(t) = A(t)x(t) + f(x(t)) \tag{2}$$

with $x(0) = x_0$, where this $K(t)$ is an $n \times n$ singular matrix, $A(t)$ is an $n \times n$ matrix, $x(t)$ is an n - state vector and *f* is an "n" vector function.

III. He's Variational Iteration Method

In this section, we briefly review the main points of the powerful method, known as the He's variational iteration method [8-14]. This method is a modification of a general Lagrange multiplier method proposed by Inokuti [14]. In the variational iteration method, the differential equation

$$L[u(t)] + N[u(t)] = g(t) \tag{3}$$

is considered, where *L* and *N* are linear and nonlinear operators, respectively and *g(t)* is an inhomogeneous term. Using the method, the correction functional

$$u_{n+1}(t) = u_n(t) + \int \lambda [L[u_n(s)] + N[\tilde{u}_n(s)] - g(s)] ds \tag{4}$$

is considered, where λ is a general Lagrange multiplier, u_n is the nth approximate solution and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$ [7–9].

In this method, first we determine the Lagrange multiplier λ that can be identified via variational theory, i.e. the multiplier should be chosen such that the correction functional is stationary, i.e. $\delta u_{n+1}(u_n(t), t) = 0$. Then the successive approximation $u_n, n \geq |0$ of the solution *u* will be obtained by using any selective initial function u_0 and the calculated Lagrange multiplier λ. Consequently $u = \lim_{n\to\infty} u_n$. It means that, by the correction functional (4) several approximations will be obtained and therefore, the exact solution emerges at the limit of the resulting successive approximations. In the next section, this method is successfully applied for solving the linear time-invariant and time-varying nonlinear singular systems.

IV. Numerical Examples for Time-Invariant Nonlinear Singular Systems

Considering the following time-invariant nonlinear singular system (Campbell [2,3], Lin and Yang [17])

$$K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } f(x(t)) = \begin{bmatrix} 0 \\ -x^2 \end{bmatrix}$$

with initial condition

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The exact solutions are

$$x_1(t) = -t$$

$$x_2(t) = \frac{t^2}{2}$$

Time t	Error Calculations			
	x_1		x_2	
	ADM Error	HVIM Error	ADM Error	HVIM Error
0	0	0	0	0
0.25	2E-07	2E-08	2E-08	2E-09
0.5	7E-07	3E-08	7E-08	3E-09
0.75	9E-07	4E-08	0.000009	4E-09
1	0.000014	5E-08	0.0000014	5E-09
1.25	0.000017	6E-08	0.0000017	6E-09
1.5	0.000019	7E-08	0.0000019	7E-09
1.75	0.000022	8E-08	0.0000022	8E-09
2	0.000026	9E-08	0.0000026	9E-09

Table 1 Error Calculations for time-invariant system for various values.

The results (approximate solutions) obtained using ADM [18] and HVIM (with step size time $\mathrm{t} = 0.25$) along with the exact solutions and its absolute errors between them are calculated and are presented in Table 1. To highlight the efficiency of the HVIM and to distinguish the effect of the errors in accordance with the exact solutions, is presented in Table 1 for selected values of " x_1 " and " x_2 ".

V. Numerical Examples for Time-Varying Nonlinear Singular Systems

Considering the time-varying nonlinear singular system of the following form (Hsiao and Wang [16] and Sepehrian and Razzaghi [21])

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & t^2 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} tx_1(t) + x_2(t) \\ \exp(t)x_1(t)x_2(t) \\ x_2(t)(x_1(t) + x_3(t)) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2t^2 \exp(-t) \\ 0 \end{bmatrix},$$

with initial condition

$$x(0) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}.$$

The exact solutions are

$$x(t) = \begin{bmatrix} 2\exp(-t)(1-t) \\ t^2 \exp(-t) \\ -2\exp(-t)(1-t) \end{bmatrix}$$

Time t	Error Calculations					
	x_1		x_2		x_3	
	ADM Error	HVIM Error	ADM Error	HVIM Error	ADM Error	HVIM Error
0	0	0	0	0	0	0
0.25	4.00E-08	4.00E-09	1.00E-08	2E-09	1.00E-07	2E-08
0.5	6.00E-08	6.00E-09	2.00E-08	3E-09	0	0
0.75	3.00E-07	3.00E-08	3.00E-08	4E-09	1.00E-08	4E-09
1	4.00E-07	4.00E-08	4.00E-08	5E-09	2.00E-08	5E-09
1.25	5.00E-07	5.00E-08	4.00E-08	6E-09	3.00E-08	6E-09
1.5	6.00E-07	6.00E-08	6.00E-08	7E-09	4.00E-08	7E-09
1.75	7.00E-07	7.00E-08	7.00E-08	8E-09	7.00E-08	8E-09
2	8.00E-07	8.00E-08	8.00E-08	9E-09	8.00E-08	9E-09

Table 2 Error Calculations for time-varying system for various values.

The results (approximate solutions) obtained using ADM [18] and HVIM (with step size time $t = 0.25$) along with the exact solutions and its absolute errors between them are calculated and are presented in Table 2. To highlight the efficiency of the HVIM and to distinguish the effect of the errors in accordance with the exact solutions, is presented in Table 2 for selected values of " x_1, x_2 " and " x_3 ".

VI. Conclusion

The obtained results of the time-invariant and time-varying nonlinear singular systems show that the He's Variational Iteration Method works well for finding the solution. From the Table 1–2, it can be observed that for most of the time intervals, the absolute error is less (almost no error) in the He's variational iteration method when compared to the ADM method [18], which yields a small error, along with the exact solutions. Hence the He's Variational Iteration Method is more suitable for studying the time-invariant and time-varying nonlinear singular systems.

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