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Oscillation Theorems for Certain Fourth Order Quasilinear Difference Equations

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Abstract: *Oscillation criteria for certain fourth order quasilinear difference equations are obtained. Examples are provided to illustrate the results. AMS Subject classification: 39A10*
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I. Introduction

Consider the fourth order quasilinear difference equation of the form

$$\Delta^2\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right)+q_n|x_{n+3}|^{\beta-1}x_{n+3}=0 \qquad (1)$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, α and β are positive constants, $\{p_n\}$ and $\{q_n\}$ are positive real sequences defined for all $n \in N(n_0) = \{n_0, n_0 + 1, ...\}$ and n_0 a nonnegative integer.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ that satisfies equation (1) for all $n \in N(n_0)$. If any four consecutive values of $\{x_n\}$ are given, then a solution $\{x_n\}$ of equation (1) can be defined recursively. A nontrivial solution of equation (1) is said to be nonoscillatory if it is either eventually positive or eventually negative and it is oscillatory otherwise.

Determining oscillation criteria for difference equations has received a great deal of attention in the last few years, see for examples [1,2] and the references cited therein. Compared to second order difference equations, the study of higher order equations, and in particular fourth order equations, has received considerably less attention, see [3-14] and the references contained therein.

In [3], the authors considered equation (1) under the following conditions

$$\sum_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}=\infty \text{ ,and } \sum_{n=n_0}^{\infty}\left(\frac{n}{p_n^{\frac{1}{\alpha}}}\right)=\infty, \qquad (2)$$

$$\sum_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}=\infty \text{ ,and } \sum_{n=n_0}^{\infty}\left(\frac{n}{p_n^{\frac{1}{\alpha}}}\right)<\infty, \qquad (3)$$

$$\sum_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}<\infty \text{ , and } \sum_{n=n_0}^{\infty}\left(\frac{n}{p_n^{\frac{1}{\alpha}}}\right)=\infty, \qquad (4)$$

$$\sum_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}<\infty \text{ , and } \sum_{n=n_0}^{\infty}\left(\frac{n}{p_n^{\frac{1}{\alpha}}}\right)<\infty, \qquad (5)$$

and discussed the asymptotic behavior of nonoscillatory solutions of equation (1). In [4,5], the authors considered equation (1) with condition (2) or (5) and discussed the oscillatory and asymptotic behavior of solutions of equation(1). Therefore, our main goal in this paper is to establish some new criteria for the oscillation of all solution of equation (1) under the condition(3). In Section 2, we present a classification of positive solutions of equation(1) and also we derived several lemmas which are useful in establishing the main results. In Section 3, we establishsome new sufficient conditions for the oscillation of all solutions of equation (1). Examples are provided in Section 4 to illustrate the results.

II. Classification of Positive solutions

In this section, we classify the positive solutions of equation (1) in terms of the signs of their differences.

Lemma 2. 1 *If $\{x_n\}$ is an eventually positive solution of equation (1), then one of the following three cases holds for all sufficiently large n:*

(I) $\Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n) > 0,\quad \Delta^2 x_n > 0, \Delta x_n > 0,$

(II) $\Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n) > 0,\quad \Delta^2 x_n < 0, \Delta x_n > 0,$

(III) $\Delta(p_n |\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n) > 0,\quad \Delta^2 x_n > 0, \Delta x_n < 0.$

Proof. The proof of this lemma can be modeled as that of Lemma 2.3 of [3], and hence the details are omitted.
Next, we present several lemmas which will be used later to prove our main results.

Lemma 2.2 *Let $\{x_n\}$ be a positive solution of equation (1). Then*

(i) $\lim_{n\to\infty} \Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n) = m \in [0,\infty)$.

Further, if $\{x_n\}$ is of type (II) or (III), then $m = 0$.

(ii) $m - \Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n) + \sum_{s=n}^{\infty} q_s x_{s+3}^{\beta} = 0$. (6)

Proof. Since $\{\Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n)\}$ is decreasing and positive, we have $m \geq 0$. Next, let $\{x_n\}$ be of type (II) or (III). If $m > 0$, then

$$\Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) \geq m$$

for $n \geq n_1 \in N(n_0)$. Summing the last inequality from n_1 to $n-1$, we obtain

$$\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) \geq \frac{mn}{2} \quad \text{for} \quad n \geq n_2$$

with large $n_2 \geq n_1$. This implies that $\{x_n\}$ must belong to type (III), since $\Delta^2 x_n > 0$ for $n \geq n_2$. We find

$$\Delta^2 x_n \geq \left(\frac{mn}{2p_n}\right)^{\frac{1}{\alpha}}, \ n \geq n_2$$

Summing the last inequality from n_2 to $n-1$, we have

$$-\Delta x_{n_2} \geq \Delta x_n - \Delta x_{n_2} \geq c_1 \sum_{s=n_2}^{n-1}\left(\frac{s}{p_s}\right)^{\frac{1}{\alpha}}$$

Where $c_1 = \left(\frac{m}{2}\right)^{\frac{1}{\alpha}}$.

This contradicts assumption (3). Hence m = 0. This proves (i).

(ii). Summing equation (1) from n to ∞ and using (i), we obtain (6). This completes the proof.

The following lemma gives a growth and decaying estimate of all positive solutions of equation (1).

Lemma 2. 3 *Let $\{x_n\}$ be a positive solution of equation (1). Then there exist positive constants c_1 and c_2 such that* $c_1\rho(n) \leq x_n \leq c_2 H(n,n_0)$ *for large n, where* $\rho(n) = \sum_{s=n}^{\infty}\frac{(s-n+1)}{p_s^{\frac{1}{\alpha}}}$ *and* $H(n,n_0) = \sum_{s=n_0}^{n-1}(n-s+1)\left(\frac{s}{p_s}\right)^{\frac{1}{\alpha}}$.

Proof. If $\{x_n\}$ belongs to type (I) or (II), then $\{x_n\}$ is monotonically increasing and hence $x_{n+3} \geq x_n \geq c_1\rho(n)$. Next, we consider solutions belonging to type (III). Since, by Lemma 2. 1, we have $p_n(\Delta^2 x_n)^{\alpha}$ is positive and monotonically increasing for all $n \geq n_1 \in N(n_0)$. Therefore, there exists a positive constant c_1 such that

$$\Delta^2 x_n \geq \left(\frac{c_1}{p_n}\right)^{\frac{1}{\alpha}}, \quad \text{for} \quad n \geq n_1. \tag{7}$$

Since $\Delta^2 x_n > 0$ and $\Delta x_n < 0$ for $n \geq n_1$, Δx_n is monotonically increasing and has a nonpositive limit. If this limit is negative , we can obtain a contradiction to the positivity of x_n. Hence $\Delta x_n \to 0$ as $n \to \infty$. Noting this fact and summing the inequality (7) from n to ∞, we obtain

$$-\Delta x_n \geq c_1^{\frac{1}{\alpha}} \sum_{s=n}^{\infty} p_s^{-\frac{1}{\alpha}}, \quad n \geq n_1.$$

since $x_n > 0$, $\lim_{n\to\infty} x_n \geq 0$ exists. A summation of the last inequality from n to ∞ gives

$$x_n \geq x_\infty + c_1^{\frac{1}{\alpha}}\rho(n), \quad n \geq n_1$$

which implies the first inequality.

Next we prove the second inequality. Since $\Delta^2\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right) < 0, n \geq n_1$ sufficiently large, summing this inequality from n_1 to $n-1$ twice, we have

$$\Delta^2 x_n \leq c_2\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}, n \geq n_2$$

where $c_2 > 0$ and $n_2 \geq n_1$. Summing this twice and using the assumption (3), we obtain the desired inequality. This completes the proof.

Next we establish some useful inequalities for positive solutions belonging to type (I) and to (III) respectively.

Lemma 2. 4 *Let $\{x_n\}$ be a positive solution of type (I) of equation(1). Then for large n*

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$$\Delta^2 x_n \geq c\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}\left[\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right)\right]^{\frac{1}{\alpha}} \qquad (8)$$

where c is some positive constant.

Proof**.** Since $\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right)$ is decreasing and $\Delta^2 x_n > 0$ for all large n , we have

$$\begin{aligned} p_n\left(\Delta^2 x_n\right)^{\alpha} &= p_{n_1}\left(\Delta^2 x_{n_1}\right)^{\alpha} + \sum_{s=n_1}^{n-1}\Delta\left(p_s\left(\Delta^2 x_s\right)^{\alpha}\right) \\ &\geq (n-n_1)\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right), \quad n \geq n_1 \end{aligned}$$

for large n_1. Clearly, this gives the desired inequality.

Lemma 2. 5 *Let $\{x_n\}$ be a positive solution of type (III) of equation (1).Then there exists a positive constant C such that the following inequalities hold for large n:*

$$\begin{aligned} x_n &\geq p_n^{\frac{1}{\alpha}}\rho(n)\Delta^2 x_n, \\ x_n^{\alpha} &\geq cn\rho^{\alpha}(n)\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right) \end{aligned} \qquad (9)$$

, (10)

and

$$x_{n+3}^{\alpha} \geq cn\rho^{\alpha}(n+3)\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right). \qquad (11)$$

Proof. By Lemma 2.1, we see that $\Delta(p_n(\Delta^2 x_n)^{\alpha}) > 0, \Delta^2 x_n > 0$, and $\Delta x_n < 0$. for $n \geq n_1 \in N(n_0)$ Since $p_n(\Delta^2 x_n)^{\alpha}$ is increasing and $\Delta x_n \to 0$ as $n \to \infty$, we have

$$\begin{aligned} -\Delta x_n = \sum_{s=n}^{\infty}\Delta^2 x_n &= \sum_{s=n}^{\infty} p_s^{\frac{1}{\alpha}}\Delta^2 x_n p_s^{\frac{-1}{\alpha}} \\ &\geq p_n^{\frac{1}{\alpha}}\Delta^2 x_n \sum_{s=n}^{\infty} p_s^{\frac{-1}{\alpha}} \end{aligned}$$

for $n \geq n_1$. Summing the last inequality from n to ∞, we obtain

$$x_n \geq \sum_{s=n}^{\infty} p_s^{\frac{1}{\alpha}}\Delta^2 x_s\left(\sum_{t=s}^{\infty} p_t^{\frac{-1}{\alpha}}\right) \geq p_n^{\frac{1}{\alpha}}\Delta^2 x_n\rho(n), \quad n \geq n_1 .$$

This implies (9).

Since $\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right)$ is decreasing, there is a positive constant c ∈ (0,1) and an integer $n_2 \in N(n_0)$ such that

$$\begin{aligned} p_{n+3}\left(\Delta^2 x_{n+3}\right)^{\alpha} \geq p_n\left(\Delta^2 x_n\right)^{\alpha} &\geq \sum_{s=n_2}^{n-1}\Delta\left(p_s\left(\Delta^2 x_s\right)^{\alpha}\right) \\ &\geq \Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right)(n-n_2) \\ &\geq cn\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right) \end{aligned} \qquad (12)$$

for sufficiently large n. Combining the inequalities (9) and (12) , we obtain

$$cn\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right)\rho^{\alpha}(n) \leq p_n\left(\Delta^2 x_n\right)^{\alpha}\rho^{\alpha}(n) \leq x_n^{\alpha},$$

or

$$cn\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right)\rho^{\alpha}(n+3) \leq p_{n+3}\left(\Delta^2 x_{n+3}\right)^{\alpha}\rho^{\alpha}(n+3) \leq x_{n+3}^{\alpha},$$

for sufficiently large n. This completes the proof.

III. Oscillation Theorems

In this section, we establish some new sufficient conditions for the oscillation of all solutions of equation (1).

Theorem 3.1 *Let $\beta \geq 1 > \alpha$. If*

$$\sum_{n=n_0}^{\infty} nq_n\rho^{\beta}(n+3) = \infty , \qquad (13)$$

then every solution of equation (1) is oscillatory.

Proof**.** Assume, to contrary, that $\{x_n\}$ is a positive solution of equation (1). Then $\{x_n\}$ falls into one of the three types (I) - (III) mentioned in Lemma 2.1. Therefore it is enough to show that in each case, we are led to a contradiction to (13).

Case(I). Let $\{x_n\}$ be a positive solution of equation (1) of type (I) for all $n \geq N \in \mathbb{N}(n_0)$. Summing the equation (1) from N to $n-1$ we have

$$\sum_{s=n_o}^{n-1} q_s x_{s+3}^{\beta} \leq \Delta\left(p_N\left(\Delta^2 x_N\right)^{\alpha}\right) < \infty \quad . \tag{14}$$

From the nature of type (I) solution, we can find a constant $c > 0$ such that $x_{n+3} \geq cn$ as n sufficiently large. This and (14) gives contradiction to (13) since $\beta \geq 1$.

Case(II). Let $\{x_n\}$ belongs to type (II) solution of equation (1). Multiply equation (1) by n and summing the resulting equation from N to $n-1$, we have

$$n\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) + p_{n+1}\left(-\Delta^2 x_{n+1}\right)^{\alpha} + \sum_{s=N}^{n-1} s q_s x_{s+3}^{\beta} = c_1 \quad ,$$

where c_1 is a constant. Letting $n \to \infty$ and using the nature of type (II) solution, we obtain

$$\sum_{n=N}^{\infty} n q_n x_{n+3}^{\beta} < \infty$$

Since $\{x_n\}$ is increasing, we have $\sum_{n=N}^{\infty} n q_n < \infty$. This clearly contradicts (13).

Case(III). Let $\{x_n\}$ belongs to type (III) solution of equation (1).From Lemmas 2.1 and 2. 2, we see that

$$\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right) = \sum_{s=n}^{\infty} q_s x_{s+3}^{\beta}, \quad n \geq N \in \mathbb{N}(n_0) \quad .$$

From the inequality (9) and the fact that $p_n(\Delta^2 x_n)^{\alpha}$ is increasing we see that

$$\begin{aligned}
\Delta\left(p_n(\Delta^2 x_n)^{\alpha}\right) &\geq \sum_{s=n}^{\infty} q_s \rho^{\beta}(s+3) p_{s+3}^{\frac{\beta}{\alpha}}\left(\Delta^2 x_{s+3}\right)^{\beta} \\
&\geq \sum_{s=n}^{\infty} q_s \rho^{\beta}(s+3) p_{s+1}^{\frac{\beta}{\alpha}}\left(\Delta^2 x_{s+1}\right)^{\beta} \\
&\geq \left(p_{n+1}(\Delta^2 x_n)^{\alpha}\right)^{\frac{\beta}{\alpha}} \sum_{s=n}^{\infty} q_s \rho^{\beta}(s+3) \quad .
\end{aligned}$$

Let $Z_n = p_n(\Delta^2 x_n)^{\alpha}$. Then

$$\Delta Z_n \geq Z_{n+1}^{\frac{\beta}{\alpha}} \sum_{s=n}^{\infty} q_s \rho^{\beta}(s+3) \quad . \tag{15}$$

For $Z_n \leq t \leq Z_{n+1}$, we have

$$\int_{Z_n}^{Z_{n+1}} \frac{dt}{t^{\frac{\beta}{\alpha}}} \geq \frac{\Delta z_n}{Z_{n+1}^{\frac{\beta}{\alpha}}} . \tag{16}$$

Using (16) in (15) and then summing the resulting inequality from N to $n-1$, we find that

$$\frac{\alpha}{\alpha-\beta}\left(Z_n^{\frac{\alpha-\beta}{\alpha}} - Z_N^{\frac{\alpha-\beta}{\alpha}}\right) \geq \sum_{s=N}^{n-1}\sum_{t=s}^{\infty} q_t \rho^{\beta}(t+3) \quad . \tag{17}$$

From $\frac{\alpha-\beta}{\alpha} < 0$ and the fact that Z_n is increasing, letting $n \to \infty$ in the last inequality (17), we obtain a contradiction to (13). This completes the proof.

To prove our next theorem, we require the additional assumption on $\{p_n\}$ which means that $\{p_n\}$ behave like a constant multiple of the function $\{n^k\}$ as n sufficiently large:

$$0 < \lim_{n\to\infty} inf\left(\frac{p_n}{n^k}\right) \leq \lim_{n\to\infty} sup\left(\frac{p_n}{n^k}\right) < \infty, \qquad \text{for some } k \in \mathbb{R} \tag{18}$$

Note that our condition (3) implies that $2\alpha < k \leq \alpha + 1$ in (18).

Theorem 3. 2 *Let $\beta < \alpha < 1$. Suppose that condition (18) holds. If*

$$\sum_{n=n_0}^{\infty} q_n \ H(n, n_0)^{\beta} = \infty \tag{19}$$

then every solution of equation (1) is oscillatory.

Proof. Assume, to contrary, that $\{x_n\}$ is a positive solution of equation (1). Then $\{x_n\}$ falls into one of the three types (I)-(III) mentioned in Lemma 2.1 . Therefore it is sufficient to prove that in each case we are led to a contradiction to (19).

Case(I). Let $\{x_n\}$ be a positive solution of equation (1) of type(I) for all $n \geq N \in \mathrm{N}(n_0)$. By Lemma 2. 4, we can assume that (8) holds with some positive constant c for $n \geq N$. Summing (8) from N to $n-1$ twice and using the fact that Δx_n is positive we see that

$$x_{n+3} \geq x_n \geq c_1 \left(\Delta(p_n(\Delta^2 x_n)^\alpha)\right)^{\frac{1}{\alpha}} H(n,n_0), \quad n \geq N_1 \geq N$$

where $c_1 > 0$ is a constant and $N_1 \geq N$ is large enough. Substituting this estimate in equation(1), we obtain

$$-\Delta^2\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) \geq c_1^\beta q_n \left(\Delta\left(p_n(\Delta^2 x_n)^\alpha\right)\right)^{\frac{\beta}{\alpha}} H^\beta(n,n_0),$$

that is ,

$$-\Delta^2\left(p_n\left(\Delta^2 x_n\right)^\alpha\right)\left(\Delta\left(p_n\left(\Delta^2 x_n\right)^\alpha\right)\right)^{\frac{-\beta}{\alpha}} \geq c_1^\beta q_n H^\beta(n,n_0). \tag{20}$$

Let $Z_n = \Delta\left(p_n\left(\Delta^2 x_n\right)^\alpha\right)$. Then for $Z_{n+1} \leq t \leq Z_n$, we have

$$\int_{Z_{n+1}}^{Z_n} \frac{dt}{t^{\frac{\beta}{\alpha}}} \geq \frac{-\Delta Z_n}{Z_n^{\frac{\beta}{\alpha}}}. \tag{21}$$

Using (21) in (20) and summing the resulting inequality from N_1 to $n-1$, we obtain

$$\begin{aligned}\frac{-\alpha}{\alpha-\beta} Z_n^{\frac{\alpha-\beta}{\alpha}} &\geq -\frac{1}{\alpha-\beta}\{Z_n^{\frac{\alpha-\beta}{\alpha}} - Z_{N_1}^{\frac{\alpha-\beta}{\alpha}}\} \\ &\geq c_1^\beta \sum_{s=N_1}^{n-1} q_s H^\beta(s,n_0).\end{aligned} \tag{22}$$

Let $n \to \infty$, we see that (22) contradicts (19).

Case(II). Let $\{x_n\}$ be a type (II) solution of equation (1). Then from the proof of Theorem 3.1 case(II), we see that condition $\sum_{n=N}^{\infty} n q_n < \infty$ holds. Since $k > 2\alpha$ and $\beta < \alpha$, the last stated condition implies

$$\sum_{n=N}^{\infty} q_n \quad H(n,n_0)^\beta < \infty$$

a contradiction to (19).

Case(III). Let $\{x_n\}$ be a type (III) solution of equation (1) for $n \geq N \in \mathrm{N}(n_0)$. Summing the equation (1) from n to ∞ and applying Lemma 2.2, we have

$$\Delta\left(p_n\left(\Delta^2 x_n\right)^\alpha\right) = \sum_{s=n}^{\infty} q_s x_{s+3}^\beta$$

for large n. From (11), we have

$$\Delta\left(p_n\left(\Delta^2 x_n\right)^\alpha\right) \geq c \sum_{s=n}^{\infty} q_s s^{\frac{\beta}{\alpha}} \rho_{s+3}^\beta \left(\Delta\left(p_s\left(\Delta^2 x_s\right)^\alpha\right)\right)^{\frac{\beta}{\alpha}}, \tag{23}$$

where c is a positive constant. Let Z_n denote the right hand side of the inequality (23). Then we find

$$\begin{aligned}-\Delta Z_n &= c n^{\frac{\beta}{\alpha}} q_n \rho^\beta(n+3)\{\Delta\left(p_n\left(\Delta^2 x_n\right)^\alpha\right)\}^{\frac{\beta}{\alpha}} \\ &\geq c n^{\frac{\beta}{\alpha}} q_n \rho^\beta(n+3) Z_n^{\frac{\beta}{\alpha}}, \quad n \geq N.\end{aligned}$$

Then following the argument used in the proof of case(III) of Theorem 3. 1 , we obtain that

$$\sum_{n=N}^{\infty} q_n n^{\frac{\beta}{\alpha}} \rho^\beta(n+3) < \infty. \tag{24}$$

From the assumption (18), we see that

$$H^\beta(n,n_0) \leq c_2 n^{\beta(2+\frac{1-k}{\alpha})} \leq c_3 n^{\frac{\beta}{\alpha}} \rho^\beta(n+3) \text{ for large } n, \tag{25}$$

where c_2 and c_3 are positive constants. From (24) and (25), we obtain a contradiction to (19). This completes the proof.

IV. Examples

In this section, we present two examples to illustrate the oscillation results.

Example 4. 1 Consider the difference equation

$$\Delta^2\left(n\left(\Delta^2 x_n\right)^{\frac{1}{3}}\right) + 4^{\frac{4}{3}}(n+1)x_{n+3}^3 = 0, \; n \geq 1. \tag{26}$$

Here $p_n = n,\ q_n = 4^{\frac{4}{3}}(n+1),\ \alpha = \frac{1}{3}$ and $\beta = 3$. It is easy to see that all conditions of Theorem 3.1 are satisfied and hence every solution of equation (26) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (26).

Example 4. 2 Consider the difference equation

$$\Delta^2\left(n\left(\Delta^2 x_n\right)^{\frac{1}{3}}\right) + 4^{\frac{4}{3}}(n+1)x_{n+3}^{\frac{1}{5}} = 0, \quad n \geq 1. \qquad (27)$$

Here $p_n = n,\ q_n = 4^{\frac{4}{3}}(n+1),\ \alpha = \frac{1}{3}$ and $\beta = \frac{1}{5}$. It is easy to see that all conditions of Theorem 3.2 are satisfied and hence every solution of equation (27) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (27).

V. Conclusion

It will be of interest to employ different techniques rather than used in this paper, and obtain criteria for the oscillation of all solutions of equation (1) when $\alpha < \beta \leq 1$ and $1 < \beta < \alpha$.

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