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Twin Edge Colourings of Wheel Graphs

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Abstract: *A proper edge colouring of a graph G with the elements of $\mathbb{Z}_k$ is said to be a twin edge k-colouring of G if the induced vertex colouring is a proper vertex colouring in which the colour of a vertex v in G is defined as the sum (in $\mathbb{Z}_k$) of the colours of the edges incident with v. Twin chromatic index of G is the minimum k for which G has a twin edge k-colouring. Twin chromatic index of W_n is determined, where W_n is the wheel graph of order n.*

Keywords: *edge colouring, vertex colouring.*

I. Introduction

Vertex labelling was introduced by Rosa [14] in 1968, which induces an *edge-distinguishing labelling* defined by subtracting labels. A vertex labelling $f : V(G)\rightarrow\{0,1,\ldots,m\}$ for a graph G of size m was called β-*valuation* by Rosa if the induced edge labelling $f\,' : E(G)\rightarrow\{1,2,\ldots,m\}$ defined by $f\,'(uv) = |f(u) - f(v)|$ was bijective. A β-valuation was called as a *graceful labelling* and a graph which possessing a graceful labelling was called a *graceful graph* [9]. A popular conjecture in graph theory, due to Anton Kotzig and Gerhard Ringel, is the following.

The Graceful Tree Conjecture: *Every nontrivial tree is graceful.*

Definition 1.1. *For a connected graph G of order $n \geq 3$, let $f : E(G)\rightarrow\mathbb{Z}_n$ be an edge labelling of G that induces a bijective function $f\,' : V(G)\rightarrow\mathbb{Z}_n$ defined by $f\,'(v)=\Sigma_{e\epsilon E v} f(e)$ for each vertex v of G, where E_v is the set of edges of G incident with a vertex v. Such a labelling f is called a modular edge-graceful labelling, while a graph possessing such a labelling is called modular edge-graceful.*

Definition 1.2. For the set ℕ of positive integers, an edge colouring c: $E(G)\rightarrow\mathbb{N}$, where adjacent edges may be coloured the same, is said to be *vertex-distinguishing* if the colouring c': $V(G)\rightarrow\mathbb{N}$ induced by c and defined by $c'(v) = \Sigma_{e\epsilon Ev}\, c(e)$ has the property that $c'(x) \neq c'(y)$ for every two distinct vertices x and y of G.

Definition 1.3. *A neighbour-distinguishing colouring* of a graph G is a colouring in which every pair of adjacent vertices of G are coloured differently. Such a colouring is more commonly called a *proper vertex colouring.* The minimum number of colours needed in a proper vertex colouring of a graph G is the *chromatic number* of G and denoted by $\chi(G)$.

Definition 1.4. For $k\epsilon\mathbb{N}$, let c: $E(G)\rightarrow\{1,2,\ldots,k\}$ be an edge colouring of G (where adjacent edges may be assigned the same colour). A vertex colouring c': $V(G) \rightarrow\mathbb{N}$ is defined where $c'(v)$ is the sum of the colours of the edges incident with v. If c' is a proper vertex colouring of G, then c is called a *neighbour-distinguishing edge colouring* of G.

The 1-2-3 Conjecture. *For every connected graph G of order at least* 3, *there exists a neighbour-distinguishing edge colouring of G using only the colours* 1, 2, 3.

Definition 1.5. In a *proper edge colouring* of a graph G, each edge of G is assigned a colour from a given set of colours where adjacent edges are coloured differently. The minimum number of colours needed in a proper edge colouring of G is called the *chromatic index* of G and is denoted by $\chi'(G)$.

Observation 1.6. *For every nonempty graph G,* $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$.(This was proved by Vizing [15]).

Definition 1.7. *Total colouring* of a graph G that assigns colours to both the vertices and edges of G so that not only the vertex colouring and edge colouring are proper but no vertex and an incident edge are assigned the same colour. The minimum number of colours required for a total colouring of G is the *total chromatic number* of G, denoted by $\chi''(G)$.

The Total Colouring Conjecture. *For every graph G,* $\chi''(G) \leq 2+\Delta(G)$.

II. Twin Chromatic Index

For a connected graph G of order at least 3, a proper edge colouring $c: E(G) \rightarrow \mathbb{Z}_k$ for some integer $k \geq 2$ is sought for which the induced vertex colouring $c': V(G) \rightarrow \mathbb{Z}_k$ defined by

$$c'(v) = \Sigma_{e \in E v}\, c(e) \text{ in } \mathbb{Z}_k,$$

(where the indicated sum is computed in $\mathbb{Z}_k$) results in a proper vertex colouring of G. We refer to such a colouring as a *twin edge k-colouring* or simply *twin edge colouring* of G. The minimum k for which G has a twin edge k-colouring is called the *twin chromatic index* of G and is denoted by $\chi'_t(G)$. Since a twin edge colouring is not only a proper edge colouring of G but induces a proper vertex colouring of G, it follows that

$$\chi'_t(G) \geq \max\{\chi(G), \chi'(G)\}.$$

Since $\max\{\chi(G), \chi'(G)\} = \chi'(G)$ except when G is a complete graph of even order, we have $\chi'_t(G) \geq \chi'(G)$ except possibly when G is a complete graph of even order.

While $\chi'_t(G)$ does not exist if G is the connected graph of order 2, every connected graph of order at least 3 has a twin edge colouring. To see this, let G be a connected graph of size $m \geq 2$. *If* $m=2$, then assign the colours 1and 2 in $\mathbb{Z}_3$ to the two edges of G. If $m \geq 3$, then assign the m elements $0,1,2,4,\ldots,2^{m-1} \in \mathbb{Z}_{2^{m-1}}$ to the m edges of G in a one-to-one manner so that the colour 0 is assigned to a pendant edge if G has such an edge. Hence the sets of edges coloured by nonzero elements in $\mathbb{Z}_2^{m-1}$ that are incident with every two adjacent vertices are distinct. Since the base 2 representations of the colours of these vertices are different, it follows that adjacent vertices are assigned distinct colours in $\mathbb{Z}_2^{m-1}$. Thus, this colouring is a twin edge colouring. This observation yields the following.

Proposition 2.1. *If G is a connected graph of order at least* 3 *and size m, then* $\chi'_t(G)$ *exists. Furthermore,* $\chi'_t(G) \leq 2^{m-1}$ *if* $m \geq 3$.

Proposition 2.2. *If* P_n *is a path of order* $n \geq 3$, *then* $\chi'_t(P_n) = 3$.

Observation 2.3. *If a connected graph G contains two adjacent vertices of degree* $\Delta(G)$, *then* $\chi'_t(G) \geq 1+\Delta(G)$.

Proposition 2.4. *If* C_n *is a cycle of order* $n \geq 3$, *then*

$$\chi'_t(C_n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3 \\ 4, & \text{if } n \not\equiv 0 \pmod 3 \text{ and } n \neq 5 \\ 5, & \text{if } n = 5 \end{cases}$$

III. Wheel Graphs

We now investigate twin edge colourings of wheel graphs W_n.

Proposition 3.1. *If* W_n *is a wheel of order* $n \geq 5$, *then* $\chi'_t(W_n) = n(n-1)/2$.

Proof: *Let* $W_n = (v_1, v_2, \ldots, v_n)$ *be a wheel of order* $n \geq 5$ *and let* v_n *be the mid vertex of the wheel, where*

$e_i = v_n v_i, \quad i = 1,2,\ldots,n-1$

$e_{n-1+i} = v_i v_{i+1}, \quad i = 1,2,\ldots,n-2$

$e_{2n-2} = v_{n-1} v_1$, since the number of edges in a wheel graph is $2n-2$.

Now $\Delta(W_n) = n-1$. From the observation 1.6, we have $\Delta(W_n) \leq \chi'(W_n) \leq \Delta(W_n)+1$. Therefore $n-1 \leq \chi'(W_n) \leq n$. *We show that* $\chi'(W_n) = n-1$. Let c be a proper edge colouring of W_n and defined as follows.

- For even n, define an edge colouring

$c: E(W_n) \rightarrow \mathbb{Z}_{n-1}$ as follows.

$$c(e_i) = \begin{cases} i, & i = 1,2,\ldots,n-1 \\ 0, & i = n, n+2, n+4, \ldots, 2n-4 \\ 1, & i = n+1, n+3, \ldots, 2n-3 \\ 3, & i = 2n-2 \end{cases}$$

- For odd n, define an edge colouring

$c : E(W_n)\rightarrow\mathbb{Z}_{n-1}$ as follows.

$$c(e_i) = \begin{cases} i, & i = 1,2,\ldots,n-1 \\ 1, & i = n,n+2,n+4,\ldots,2n-4 \\ 0, & i = n+1,n+3,\ldots,2n-3 \\ 3, & i = 2n-2 \end{cases}$$

Thus c is a (n-1)-edge colouring. Hence $\chi'(W_n) = n-1$.It remains to show that W_n has a twin edge $n(n - 1)/2$ colouring. A colouring $c : E(W_n)\rightarrow\mathbb{Z}_{n-1}$ for both even and odd n is defined as above.

- If n is even, then

$$c'(v_i) = \begin{cases} 4, & \text{if } i = 1 \\ i+1, & \text{if } i = 2,3,\ldots,n-2 \\ n+3, & \text{if } i = n-1 \\ n(n-1)/2, & \text{if } i = n \end{cases}$$

For example, if $n = 6$, then $(c(e_1),c(e_2),\ldots,c(e_{10})) = (1,2,3,4,5,0,1,0,1,3)$ and $(c'(v_1),c'(v_2),\ldots,c'(v_6)) = (4,3,4,5,9,15)$

- If n is odd, then

$$c'(v_i) = \begin{cases} 4, & \text{if } i = 1 \\ i+1, & \text{if } i = 2,3,\ldots,n-2 \\ n+3, & \text{if } i = n-1 \\ n(n-1)/2, & \text{if } i = n \end{cases}$$

For example, if $n = 7$, then $(c(e_1),c(e_2),\ldots,c(e_{12})) = (1,2,3,4,5,6,0,1,0,1,0,3)$ and $(c'(v_1),c'(v_2),\ldots,c'(v_7)) = (4,3,4,5,6,9,21)$. Hence $\chi'_t(W_n) = n(n - 1)/2$.

IV. Conclusion

In this paper we determined the twin chromatic index of a wheel graph W_n. We may proceed this concept of finding twin chromatic index for some more graphs.

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