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On Double Cosets in Free Groups

Rita Gitik and Eliyahu Rips

ABSTRACT. This paper had been written more than twenty years ago. It exhibits a very short geometric proof of the following fact: for any finitely generated subgroups H and K of a free group F, and for any $g \in F$ the double coset HgK is closed in the profinite topology of F. An updated bibliography is included.

A well-known theorem of M. Hall [4] states (in different language) that any finitely generated subgroup of a free group is closed in the profinite topology. We show that a slight modification of its proof [9] yields a stronger result:

Theorem 1. *For any finitely generated subgroups H and K of a free group F, and for any $g \in F$ the double coset HgK is closed in the profinite topology of F.*

A free group $F = < X >$ can be viewed as the fundamental group of a wedge W of $|X|$ oriented circles labeled by elements of X. Subgroups of F correspond bijectively to based covering spaces of W, and any covering space of W is a graph which inherits orientation and labeling of its edges from W.

Let $X_0 \subset X$ and let Γ be a subgraph of a covering of W. An X_0-component of Γ is a maximal connected subgraph of Γ with all its edges labeled by elements of X_0.

Lemma 1. *Let Γ be a subgraph of a covering of W such that Γ has finitely many vertices. There exists an embedding of Γ in a covering Γ' of W such that Γ and Γ' have the same vertices, and for any $X_0 \subset X$ distinct X_0-components of Γ remain distinct in Γ'.*

Proof. We give an algorithm for constructing Γ' by adding edges to Γ in a unique way. For any vertex v of Γ and for any $x \in X$ the number of edges labeled with x having an endpoint at v is either $0, 1$ or 2. If the number is 0, we add an edge labeled with x with both endpoints at v. If the number is 1 or 2, let p_x be the maximal path consisting of edges labeled only with x and with an endpoint at v. If p_x has both endpoints at v we do nothing. Otherwise we add to Γ an edge labeled with x connecting the endpoints of p_x. It is clear that the projection from Γ to W extends uniquely to a covering map from Γ' to W. $\square$

Remark 1. *Let H be a finitely generated subgroup of F, and let $f \in F \setminus H$. Let Γ be the minimal connected subgraph of the covering C of W corresponding to H which contains the core of C (cf. [9]) and the path p beginning at the basepoint v_0 of C whose projection in W represents f. Embed Γ in a covering Γ' as in the lemma. Then as Γ has finitely many vertices, so does Γ', therefore the subgroup M of F corresponding to Γ' has finite index in F. As p is not a closed path in Γ, it remains not closed in Γ', hence $f \notin M$. As Γ is a subgraph of Γ', H is a subgroup of M, proving M. Hall's theorem (cf. [9]).*

Proof. Proof of the theorem. As $HgK = H(gKg^{-1})g$, it is enough to consider the case $g = 1$.

By an observation due to P.Kropholler, we can replace F by a subgroup of finite index (cf. [7]), so we can assume that K is a free factor of F, $F = K * L$. Let X_1 and X_2 be sets of free generators of K and L respectively, then $X = X_1 \cup X_2$ is a set of free generators of F. Let $f \in F, f \notin HK$. Our goal is to construct a subgroup M of finite index in F such that $Mf \cap HK = \emptyset$.

Let Γ, Γ' and M be as in the remark. As $f \notin HK$, the X_1-component of v_0 in Γ does not contain the endpoint of p, therefore the lemma implies that Γ' has the same property. But the condition $Mf \cap HK = \emptyset$ is equivalent to the condition that the endpoint of p does not belong to the X_1-component of v_0, proving the theorem. □

Remark 2. (1) *The first published proof of the theorem is due to G.A. Niblo* [7]*, who found a much more simple and elegant argument than the original proof of the authors* [3]*.*

(2) *A more general result saying that for any finitely generated subgroups $H_1, \cdots, H_n$ of a free group F the set $H_1 \cdots H_n$ is closed in the profinite topology on F was obtained by L. Ribes and P.A. Zalesskii* [8]*, by K. Henckell, S.T. Margolis, J.E. Pin and J. Rhodes* [5] *and by B. Steinberg in* [10]*.*

(3) *T. Coulbois in* [1] *proved that property RZ_n is closed under free products, where a group G is said to have property RZ_n if for any n finitely generated subgroups $H_1, \cdots H_n$ of G, the set $H_1 \cdots H_n$ is closed in the profinite topology of G.*

(4) *The first author showed in* [2] *that if H and K are quasiconvex subgroups of a negatively curved LERF group G and H is malnormal in G then the double coset KH is closed in the profinite topology of G.*

(5) *A. Minasyan generalized that result in* [6] *showing that a product of any finite number of quasiconvex subgroups of a negatively curved GFERF group G is closed in the profinite topology of G.(A negatively curved group G is GFERF if all its quasiconvex subgroups are closed in the profinite topology of G.)*

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Department of Mathematics, University of Michigan, Ann Atbor, MI, 48109
E-mail address: ritagtk@umich.edu

Institute of Mathematics, Hebrew University, Jerusalem, 91904, Israel