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On the Asymptotic Behavior of Solutions for a Class of Second Order Nonlinear Difference Equations

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Abstract: *We study the asymptotic behaviour of solutions for a class of second order nonlinear difference equations. Using Bihari's inequality, we obtain conditions under which all solutions are asymptotic to $an+b$ as $n\to\infty$, where a and b are real constants.*

Keywords: *Asymptotic behaviour, Bihari's inequality, second order nonlinear difference equations.*

I. Introduction

In this paper, we are concerned with the nonlinear difference equation

$$\Delta^2 u(n) + f(n,u,\Delta u) = 0 \tag{1}$$

which is used in modeling of a large number of physical systems. Here Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$ where $n \in N_0 = \{n_0, n_0+1, n_0+2, \cdots\}$ and $n_0 \in Z^+$.

By a solution of equation (1), we mean a sequence of real numbers which is defined for $n \ge n_0 \in N_0$ and which satisfies equation (1). A solution $\{x(n)\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and it is nonoscillatory otherwise. Asymptotic behavior of solutions of second order difference equations have been studied by many authors, for example see [5, 7-16] and the references cited there in. For a general background on difference equations see the monographs [1, 2, 17].

II. Main Results

We begin with the following result.

Theorem 2.1. *Suppose that the function $f(n,u,v)$ satisfies the following conditions.*

(i) f is continuous in $D = \{(n,u,v)/n \in N_0, u,v \in R\}$, where $n_0 \ge 1$;

(ii) There exists two sequences $h(n)$ and $g(n)$ such that

$$|f(n,u,v)| \le h(n) g\left(\frac{|u|}{n}\right)|v|,$$

where for $s > 0$ the sequence $g(s)$ is positive and nondecreasing,

$$\sum_{n_0}^{\infty} h(s) < \infty$$

and if we denote

$$G(n) = \sum_{n_0}^{n-1} \frac{\Delta z(n)}{g(z(n))}$$

and $G(+\infty) = +\infty$. Then every solution $u(n)$ of equation (1) is asymptotic to $an+b$ where a and b are real constants.

Proof. It follows from (i) by standard existence Theorem (see, for example,[1]) that equation (1) has solution $u(n)$ corresponding to the initial data $u(n_0) = c_1$ and $\Delta u(n_0) = c_2$.

Summing two times (1) from n_0 to $n-1$, we get for $n \ge n_0$

$$\Delta u(n) = c_2 - \sum_{n_0}^{n-1} f(s,u(s),\Delta u(s)) \tag{2}$$

and

$$u(n) = c_2(n-1-n_0) + c_1 - \sum_{n_0}^{n-1} (n-1-s) f(s,u(s),\Delta u(s)). \tag{3}$$

It follows from (2) and (3) for $n \ge n_0$

$$|\Delta u(n)| \le |c_2| + \sum_{n_0}^{n-1} |f(s,u(s),\Delta u(s))|$$

$$|u(n)| \le |c_1| + |c_2|n + n\sum_{n_0}^{n-1} |f(s,u(s),\Delta u(s))|.$$

So applying (ii) in the above inequalities, we get for $n \ge n_0$

$$|\Delta u(n)| \le |c_2| + \sum_{n_0}^{n-1} h(s)g\left(\frac{|u(s)|}{s}\right)|\Delta u(s)| \tag{4}$$

$$\left|\frac{u(n)}{n}\right| \le |c_1| + |c_2| + \sum_{n_0}^{n-1} h(s)g\left(\frac{|u(s)|}{s}\right)|\Delta u(s)| \tag{5}$$

Denote by $z(n)$, the right hand side of inequality (5)

$$z(n) = |c_1| + |c_2| + \sum_{n_0}^{n-1} h(s)g\left(\frac{|u(s)|}{s}\right)|\Delta u(s)|$$

Hence (4) and (5) imply that

$$|\Delta u(n)| \le z(n) \quad and \quad \frac{|u(n)|}{n} \le z(n) \tag{6}$$

The sequence $g(s)$ is nondecrasing for $s > 0$ and hence we get by (6)

$$g\left(\frac{|u(n)|}{n}\right) \le g(z(n))$$

so we conclude that for $n \ge n_0$

$$|z(n)| \le 1 + |c_1| + |c_2| + \sum_{n_0}^{n-1} h(s)g(z(s))z(s). \tag{7}$$

Applying discrete Bihari's inequality [17] to (7), we get for $n \ge n_0$

$$|z(n)| \le G^{-1}\left(G(1+|c_1|) + |c_2| + \sum_{n_0}^{n-1} h(s)\right)$$

where G is the solution of $G(z(n)) = \sum_{n_0}^{n-1} \frac{\Delta z(n)}{g(z(n))}$ and $G^{-1}(z(n))$ is the inverse function of $G(z(n))$ which is defined on $N_1 = \left\{ n \in N_{n_0}^{+} \mid \sum_{n_0}^{n-1} n(s) \le G(\infty) - G(x_0) \right\}$.

Now put

$$K = G(1 + |c_1| + |c_2|) + \sum_{n_0}^{n-1} h(s) < \infty.$$

Since G^{-1} is increasing, we get

$$z(n) \le G^{-1}(k) < \infty,$$

so it follows from (6) that

$$\frac{|u(n)|}{n} \le G^{-1}(k) \quad and \quad |\Delta u(n)| \le G^{-1}(k).$$

By (ii) we have

$$\sum_{n_0}^{n-1} |f(s,u(s),\Delta u(s))| \le \sum_{n_0}^{n-1} h(s)g\left(\frac{|u(s)|}{s}\right)|\Delta u(s)|$$

$$\le |c_1| + |c_2| + \sum_{n_0}^{n-1} h(s) g\left(\frac{|u(s)|}{s}\right) |\Delta u(s)|$$
$$= z(n)$$
$$\le G^{-1}(k)$$

therefore

$$\sum_{n_0}^{\infty} |f(s, u(s), \Delta u(s))|$$

exists as well as there exists an $a \in R$ such that

$$\lim_{n\to\infty} \Delta u(n) = a.$$

Further, in the same way as in [1, 2, 17] we can ensure that there exists a solutions with the property

$$\lim_{n\to\infty} \Delta u(n) \neq 0.$$

Finally, by the L'Hospital's rule, we conclude that

$$\lim_{n\to\infty} \frac{|u(n)|}{n} = \lim_{n\to\infty} \Delta u(n) = a$$

and thus there exists a $b \in R$ such that

$$\lim_{n\to\infty} \left(u(n) - (an + b)\right) = 0$$

so the proof is now complete. ■

Example 2.1. *Consider the nonlinear difference equation*

$$\Delta^2 u(n) - \frac{2n}{n+2}[u(n) - (an+b)][\Delta u(n) - a] = 0 \qquad (E_1)$$

where $\sum h(n) = \sum_{n_0}^{\infty} \frac{2n}{n+2} < \infty$ and $G(\infty) = \sum_{n_0}^{\infty} h(s) z(s) = \infty$. Therefore equation (E_1) satisfies all conditions of Theorem 2.1. Therefore we deduce that for any solution $u(n)$ of equation (E_1) there exist real a, b such that $u(n) = an + b - \frac{1}{n}$.

Theorem 2.2. *Suppose that apart from assumption (i') of Theorem 2.1 the function* $f(n,u,v)$ *satisfies the following condition, (ii') there exists two positive sequences* $h(n), g(n)$ *such that*

$$|f(n,u,v)| \le h(n) \frac{|u|}{n} g(|v|)$$

where for $s > 0$, the sequence $g(s)$ is positive and nondecrasing,

$$\sum_{n_0}^{\infty} h(s) < \infty$$

and if we denote

$$G(n) = \sum_{n_0}^{n-1} \frac{\Delta z(n)}{g(z(n))}$$

then $G(+\infty) = +\infty$. Then every solution $u(n)$ of equation (1) is asymptotic to $an+b$ where a, b are real constants.

The proof of the theorem is analogous to that of Theorem 2.1 and thus it is omitted.

Corollary 2.1. *Consider the equation*

$$\Delta^2 u(n) + a(n) u(n) = 0 \qquad (8)$$

where $\sum^{\infty} na(n) < \infty$. Then $\lim_{n\to\infty} \Delta u(n)$ exists and the general solution of equation (8) is asymptotic to $d_0 + d_1 n$ as $n \to \infty$ where d_1 may be zero or d_0 may be zero but not both simultaneously.

Proof. The conclusion of corollary follows from Theorem 2.2 with $h(n) = na(n)$ and $g(z(n)) = 1$. ■

Example 2.2. *Consider the nonlinear difference equation*

$$\Delta^2 u(n) + \frac{n^2(n+2)^2}{(2n+1)(n^3+6n^2+12n+4)}\left[u(n)-(an+b)\right]\left[\Delta u(n)-a\right] = 0, \quad n \ge 1 \qquad (E_2)$$

All conditions of Theorem 2.2 are satisfied. Hence it follows from Theorem 2.2 that for any solution $u(n)$ of equation (E_2), there exist real constant a,b such that $u(n) = an + b + \frac{1}{n^2}$ as $n \to \infty$.

Finally, arguing in the same way as in Theorem 2.1, we can also prove the following general result which with $g_2(z(n)) = 1$ gives us exactly Theorem 2.1 and with $g_1(z(n)) = 1$ gives us exactly Theorem 2.2.

Theorem 2.3. *Suppose that apart from the assumption (i") of Theorem 2.1 the function* $f(n,u,v)$ *satisfies the following condition (ii") there exists positive sequences* $h(n), g_1(n)$ *and* $g_2(n)$ *such that*

$$|f(n,u,v)| \le h(n) g_1\left(\frac{u(n)}{n}\right) g_2(|v|)$$

where for $s > 0$, the sequence $g_1(s)$ and $g_2(s)$ are positive and nondecreasing,

$$\sum_{n_0}^{\infty} h(s) < \infty$$

and if we denote

$$G(n) = \sum_{n_0}^{n-1} \frac{\Delta z(n)}{g_1(z(n)) g_2(z(n))}$$

then $G(+\infty) = +\infty$. Then every solution $u(n)$ of equation (1) is asymptotic to $an+b$ where a,b are real constants.

Corollary 2.2. *Consider the well-known Emden-Fowler equation*

$$\Delta^2 u(n) + n^{\sigma}(u(n))^m = 0 \qquad (9)$$

with $0 < m < 1$ and $\sigma + 2 < 0$. Then $\lim_{n\to\infty} \Delta u(n)$ exists and all solutions of equation (9) is asymptotic to $d_0 + d_1 n$ as $n \to \infty$.

Proof. The conclusion of corollary follows from Theorem 2.3 with $h(n) = n^{\sigma+m}$, $g_1(s) = s$ and $g_2(s) = 1$. ■

Example 2.3. *Consider the nonlinear difference equation*

$$\Delta^2 u(n) - \frac{(2n+1)^k (n+3)}{(n(n+1)(n+2))^k (n^2+3n+1)^{1-k}} \left(\frac{u(n)}{u(n)+n}\right)^k (\Delta u(n))^{1-k} = 0 \qquad (E_3)$$

where $k \in (0,1)$ and $a(n) = \frac{(2n+1)^k (n+3)}{(n(n+1)(n+2))^k (n^2+3n+1)^{1-k}}$. All conditions of Theorem 2.3 are satisfied. Then for any solution $u(n)$ of equation (E_3), the existence of real numbers a,b such that $u(n) = an + b + o(n)$ as $n \to \infty$.
Observe that

$$u(n) = \frac{n^2}{n+1}$$

is the one such solution of equation (E_3) with the initial data $u(n) = \frac{1}{2}$ and $\Delta u(1) = \frac{5}{6} \ne 0$.

Example 2.4. *Consider the sublinear difference equation*

$$\Delta^2 u(n) + \frac{2}{n^2 \Gamma n - 1}(u(n))^{1/2} = 0, \quad n \ge 2 \qquad (E_4)$$

where $0 < m = \frac{1}{2} < 1$ and $\sigma + 2 < 0$. All conditions of Corollary 2.2 are satisfied and all solutions of equation (9) is asymptotic to $d_0 + d_1 n$ as $n \to \infty$.

Remark 2.1. *It is important to note that by Theorem 2.1-2.3 and Corollaries 2.1 and 2.2 all solutions of equations (1), (8) and (9) respectively are asymptotic to* $an+b$ *as* $n \to \infty$ *though the restrictions on the function* $f(n,u,\Delta u)$ *may seem to be artificial. It is possible to relax these assumptions for a certain rather wide class of sequences though the price to be paid for it is the desired asymptotic behavior only for a part of solutions of equation (1) with initial data satisfying a certain estimate.*

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References

[1]. Agarwal. R.P., Difference Equations and Inequalities Theory, Method and Applications, *Marcel Dekker, New York,* 1992.
[2]. Agarwal. R.P and Wong P.J.Y., Advanced Topics in Difference Equations, *Kluwer Academic Publishers Group, Dordrecht,* 1997.
[3]. Bellman. R., Stability Theory of Differential Equations, *McGraw-Hill, London,* 1953.
[4]. Bihari. I., A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta. Math. Acad. Sci. Hung.,* **7** (1956), 83-94.
[5]. Cheng. S.S., Li. H.J, and Patula. W.T., Bounded and zero convergent solutions of second order difference equations, *J. Math. Anal. Appl.* **141** (1989), 463-484.
[6]. Cronin. J, Differential Equations, Introduction and Qualitative Theory, *Dekker, New York,* 1980.
[7]. Dai. B.X., and Huang. L. H., Asymptotic behavior of solutions for a class of nonlinear difference equations, *Comput. Math. Appl.,* **36** (1998), 23-30.
[8]. He. X.Z, Oscillatory and asymptotic behavior of second order nonlinear difference equations, *J. Math. Anal. Appl.* **175** (1993), 482-498.
[9]. Hooker. J.W and Patula. W.T., A second order nonlinear difference equation. oscillation and asymptotic behavior, *J. Math. Anal. Appl.,* **91** (1983), 9-29.
[10]. Medina. R., Asymptotic behavior of solutions of second order difference equations, *J. Comput. Appl. Math.,* **64** (1995), 233-246.
[11]. Thandapani. E., Asymptotic and oscillatory behavior of solutions of a second order nonlinear neutral delay difference equations, *Riv. Mat. Univ. Varma* **5** (1992), 105-113.
[12]. Thandapani.E., Sundar. P., Graef. J.R and Spikes. P.W., Asymptotic properties of solutions of nonlinear second order neutral delay difference equations, *Dynamic. Syst. Appl.* **41**(1995), 125-136.
[13]. Thandapani. E and Marian.S.I, The asymptotic behavior of solutions of nonlinear second order difference equations, *Appl. Math. Lett.,* **4** (2001), 611-616.
[14]. Wang. J., Meng. F and Li.J., Asymptotic behavior of solutions of second order nonlinear difference equations, *Kodai Math. J.,* **19** (1996), 200-206.
[15]. Wei. G, Asymptotic behavior results for nonlinear neutral delay difference equations, *Appl. Math. Comp.,* **217** (2011), 7184-7190.
[16]. Zhang. B.G and Zhou. Y., Oscillation and nonoscillation for second order linear difference equations, *Comp. math. Appl.* **39** (2000), 1-7.
[17]. Lakshikantham. V and Trigiante, Theory of Difference Equations Numerical Methods and Applications, *Academic Press, INC, New York,* 1988.