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On Non-Newtonian Real Number Series

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Abstract: *In this study, non-Newtonian real number series were introduced and their convergence conditions were investigated.*

I. Introduction

Non-Newtonian calculus was created by Michael Grossmann and Robert Katz between 1967-1970 years. They have firstly identified classical, geometric, harmonic and quadratic calculus, then bigeometric, biharmonic and biquadratic calculus. They completed the book that make up the basic framework of the non-Newtonian calculus in 1972 year [1]. In a study conducted by James R. Megginniss, it was used the non-Newtonian calculus to create probability theory adapted to human behavior and to make a selection [2]. Janne Grossmann worked on derivative and integral in the meta-calculus [3]. In a study made by Michael Grossmann, it was examined the derivative as independent from measure [4]. M. Rybaczuk and M. Stoppel have used the bigeometric calculus in the fractals and material science [5]. Dorota Janiszewski has worked on the multiplicative Runge-Kutta methods [6]. Agarmirza E. Bashirov, Emine Mısırlı and Ali Özyapıcı have done the studies on the geometric calculus and its applications [7]. In a study made by Ali Uzer, it was examined the multiplicative type of complex calculus as an alternative to the classic calculus [8]. Agarmirza E. Bashirov and Mustafa Rıza worked on the complex multiplicative calculus [9]. Cengiz Türkmen and Feyzi Başar have obtained some basic results about the sequence sets in the geometric calculus [10]. Luc Florak and Hans van Assen used the non-Newtonian calculus in the biomedical image analysis [11]. In a study made by Ahmet F. Çakmak and Feyzi Başar, some new results on the sequence spaces based on the non-Newtonian calculus have been found [12]. Sabiha Tekin and Feyzi Başar worked on specific complex sequence spaces [13]. Diana Andrada Filip and Cyrille Piatecki used the non-Newtonian calculus to reaffirm and analyse the neoclassical growth model(Solow-Swan) in the economics [14]. In a study made by Cenap Duyar, Birsen Sağır and Oğuz Oğur, some basic topological properties on the non-Newtonian real line were obtained [15].

In the light of these studies made, it has emerged the need to examine the non-Newtonian real number series and their convergence properties. In this study, we introduce the non-Newtonian real number series and give and prove some convergence tests for them.

II. General Informations

2.1. α-Arithmetic.

Definition 1. *A generator is a one-to-one function α, whose domain is $\mathbb{R}$, the set of all real numbers, and whose range is a subset of $\mathbb{R}$. We denote by $\mathbb{R}(N)$, called Non-Newtonian real line, the range of generator α. Non-Newtonian arithmetic operations on $\mathbb{R}(N)$ are represented as follows([1],[2],[3],[4],[6]):*

$$\begin{array}{ll} \alpha-addition & x\dot{+}y=\alpha\left(\alpha^{-1}(x)+\alpha^{-1}(y)\right) \\ \alpha-subtraction & x\dot{-}y=\alpha\left(\alpha^{-1}(x)-\alpha^{-1}(y)\right) \\ \alpha-multiplication & x\dot{\times}y=\alpha\left(\alpha^{-1}(x)\times\alpha^{-1}(y)\right) \\ \alpha-division & x\dot{/}y=\alpha\left(\alpha^{-1}(x)/\alpha^{-1}(y)\right) \\ \alpha-order & x\dot{<}y\left(x\dot{\leq}y\right)\Leftrightarrow\alpha^{-1}(x)<\alpha^{-1}(y)\left(\alpha^{-1}(x)\leq\alpha^{-1}(y)\right). \end{array}$$

In this case, it is said to generate an $\alpha-$arithmetic of α. For example, the identity function I generate the calassical arithmetic and the exponential function exp generate geometric arithmetic. Each generator generates a single arithmetic, contrarily each arithmetic is generated by a single generator [1].

Definition 2. *A α-positive number is a number x with $\dot{0}<x$, similarly a α-negative number is a number x with $\dot{0}>x$. α-zero and α-one numbers are denoted by $\dot{0}=\alpha(0)$ and $\dot{1}=\alpha(1)$ respectively. α-integers is obtained sequentially by adding $\dot{1}$ to $\dot{0}$ and by subtracting $\dot{1}$ from $\dot{0}$. α-integers are as follows:*

$$...,\alpha(-2),\alpha(-1),\alpha(0),\alpha(1),\alpha(2),...\,.$$

Each integer n according to α-arithmetic is denoted by $\dot{n}=\alpha(n)$. If $\dot{n}$ is an α-positive integer, then it is n times sum of $\dot{1}$[1].

Definition 3. *α-absolute value of a number $x\in\mathbb{R}(N)$ is defined by*

$$|x|_{\alpha}=\begin{cases} x & ,if\ \ x\dot{>}\dot{0} \\ \dot{0} & ,if\ \ x=\dot{0} \\ \dot{0}\dot{-}x, & if\ \ x\dot{<}\dot{0} \end{cases}.$$

This value is equivalent to the expression $\alpha\left(\left|\alpha^{-1}(x)\right|\right)$ [1].

Definition 4. *A closed α-interval on $\mathbb{R}(N)$ is represented by*

$$\begin{aligned} [a,b]^{\cdot} &= \left\{x\in\mathbb{R}(N):\ a\dot{\leq}x\dot{\leq}b\right\} \\ &= \left\{x\in\mathbb{R}(N):\alpha^{-1}(a)\leq\alpha^{-1}(x)\dot{\leq}\ \alpha^{-1}(b)\right\}=\alpha\left(\left[\alpha^{-1}(a),\alpha^{-1}(b)\right]\right), \end{aligned}$$

similarly an open α-interval $(a,b)^{\cdot}$ can be represented . It is said that an α-interval has α-lenght $b\dot{-}a$ [[2], [3]].

Let $\{u_n\}$ be an infinite sequence of the numbers in $\mathbb{R}(N)$. If each open α-interval containing an element u includes all elements outside of a finite number of the elements of the sequence $\{u_n\}$, then it is said that the sequence $\{u_n\}$ is α-converge to u and the element u is called as α-limit of the sequence $\{u_n\}$. This convergence becomes the classic convergence if $\alpha=I$. Classic and geometric convergence are equivalent in the sense that a positive number sequence $\{\bar{p}_n\}$ converges as geometric to a positive number $\bar{p}$ iff $\{p_n\}$ converges as classic to p [1]. Throughout this study, the symbol $\mathbb{N}$ denotes all positive integers.

2.2. ***-Calculus.** Let α and β be arbitrarily chosen generators which image the set $\mathbb{R}$ to A and B respectively. *-Calculus is defined as an ordered pair of the arithmetics (α-arithmetic, β-arithmetic) and the following notation is used:

	α-arithmetic	β-arithmetic
Universe	$A\,(=\mathbb{R}\,(N))$	$B\,(=\mathbb{R}\,(N))$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Ordering	$\dot{<}$	$\ddot{<}$

α-arithmetic is used on inputs and β-arithmetic is used on outputs. In particular, the changes in inputs and outputs is measured by α-arithmetic and β-arithmetic, respectively. The operators in *-calculus is applied to functions which inputs and outputs belong to A and B, respectively.

Definition 5. *The *-limit in a point $a \in A$ of a function f is one and only one number b in the set B, β-converged by an outputs $\{f(a_n)\}$ for all infinite sequence $\{a_n\}$, α-converged to a and its terms is different than a. In this case*

$$* - \lim_{x \to a} f(x) = b$$

is written [1].

Definition 6. *A function f is *-continuous in a point a of A iff this point a is an input for f and $* - \lim_{x \to a} f(x) = f(a)$ [1].*

The Results and Discussion

Definition 7. *An infinite sum*

$$a_1 \dot{+} a_2 \dot{+} \ldots \dot{+} a_n \dot{+} \ldots = {}_{\alpha}\sum_{n=1}^{\infty} a_n = {}_{N}\sum_{n=1}^{\infty} a_n$$

*is called the non*Newtonian real number series or α-series. If ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is a non-Newtonian real number series, then a sequence $\{S_m\}$ with the general term $S_m = {}_{\alpha}\sum_{n=1}^{m} a_n$ is called as the non-Newtonian partial sums sequence of the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$. If the sequence $\{S_m\}$ is α-convergent, then it is said that the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is α-convergent. If ${}^{\alpha}\lim_{m\to\infty} S_m = S$, then it is written ${}_{\alpha}\sum_{n=1}^{\infty} a_n = S$. If the limit ${}^{\alpha}\lim_{m\to\infty} S_m$*

is not or equal to $\dot{-}\infty$ *or* $\dot{+}\infty$, *then it is said that the series* ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ *is* α*-divergent.*

Given a series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ *and a positive integer* i *with* $i \geq 2$, *evidently it is written*

$$ {}_{\alpha}\sum_{n=1}^{\infty} a_n = {}_{\alpha}\sum_{n=1}^{i-1} a_n \dot{+} {}_{\alpha}\sum_{n=i}^{\infty} a_n. $$

Thus, the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ *is convergent iff the series* ${}_{\alpha}\sum_{n=i}^{\infty} a_n$ *is convergent, i.e removing the finite number of terms of the start does not change series' character.*

Example 1. *The equality* ${}_{\alpha}\sum_{n=1}^{\infty} \frac{\dot{1}}{\dot{2}^{(n-1)_{\alpha}}}\alpha = \dot{2}$ *holds:*

If $\{S_m\}$ is partial sums sequence of given series, then

$$
\begin{aligned}
S_1 &= \dot{2} \dot{-} \frac{\dot{1}}{\dot{2}^{(1-1)_{\alpha}}}\alpha \\
S_2 &= \dot{1} \dot{+} \frac{\dot{1}}{\dot{2}}\alpha = \alpha\left(\dot{1} + \alpha^{-1}\left(\frac{\dot{1}}{\dot{2}}\alpha\right)\right) = \alpha\left(1 + \frac{1}{2}\right) = \alpha\left(\frac{3}{2}\right) \\
&= \alpha\left(\frac{\alpha^{-1}\left(\dot{3}\right)}{\alpha^{-1}\left(\dot{2}\right)}\right) = \frac{\dot{3}}{\dot{2}}\alpha = \dot{2} \dot{-} \frac{\dot{1}}{\dot{2}^{(2-1)_{\alpha}}}\alpha \\
S_3 &= \dot{1} \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{+} \frac{\dot{1}}{\dot{4}}\alpha = S_2 \dot{+} \frac{\dot{1}}{\dot{4}}\alpha = \frac{\dot{7}}{\dot{4}}\alpha = \dot{2} \dot{-} \frac{\dot{1}}{\dot{2}^{(3-1)_{\alpha}}}\alpha \\
&\;\;\vdots \\
S_m &= \dot{2} \dot{-} \frac{\dot{1}}{\dot{2}^{(m-1)_{\alpha}}}\alpha.
\end{aligned}
$$

In this case, for all $m \in \mathbb{N}$

$$
\begin{aligned}
\left|S_m \dot{-} \dot{2}\right|_{\alpha} &= \left|\left(\dot{2} \dot{-} \frac{\dot{1}}{\dot{2}^{(m-1)_{\alpha}}}\alpha\right) \dot{-} \dot{2}\right|_{\alpha} = \alpha\left(\left|\alpha^{-1}\left(\dot{2} \dot{-} \frac{\dot{1}}{\dot{2}^{(m-1)_{\alpha}}}\alpha\right) - 2\right|_{\alpha}\right) \\
&= \alpha\left(\left|\left(2 - \frac{1}{2^{m-1}}\right) - 2\right|\right) = \alpha\left(\left|\frac{1}{2^{m-1}}\right|\right) = \alpha\left(\frac{1}{2^{m-1}}\right).
\end{aligned}
$$

Givem now any $\varepsilon \dot{>} \dot{0}$. We have

$$\left|\frac{1}{2^{m-1}} - 0\right| < \frac{1}{2^{m_0-1}} < \frac{1}{(1/\alpha^{-1}(\varepsilon))} = \alpha^{-1}(\varepsilon)$$

and thus

$$\left|S_m \dot{-} \dot{2}\right|_\alpha = \alpha\left(\frac{1}{2^{m-1}}\right) < \alpha\left(\alpha^{-1}(\varepsilon)\right) = \varepsilon$$

for all $m > m_0$, wherever $m_0 = \left[\left|\log_2 \frac{1}{\alpha^{-1}(\varepsilon)} + 1\right|\right] + 1 > \log_2 \frac{1}{\alpha^{-1}(\varepsilon)} + 1$.

Example 2. *(Non − Newtonian harmonic series). Shows that* ${}_\alpha\sum_{n=1}^{\infty} \frac{\dot{1}}{\dot{n}}\alpha$*:*

Let (S_k) be the partial sums sequence of given series. Then, for $k \in \mathbb{N}$

$$\begin{aligned} S_2 &= \dot{1} \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \\ S_4 &= \dot{1} \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{+} \left(\frac{\dot{1}}{\dot{3}}\alpha \dot{+} \frac{\dot{1}}{\dot{4}}\alpha\right) \\ S_8 &= \dot{1} \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{+} \left(\frac{\dot{1}}{\dot{3}}\alpha \dot{+} \frac{\dot{1}}{\dot{4}}\alpha\right) \dot{+} \left(\frac{\dot{1}}{\dot{5}}\alpha \dot{+} \frac{\dot{1}}{\dot{6}}\alpha \dot{+} \frac{\dot{1}}{\dot{7}}\alpha \dot{+} \frac{\dot{1}}{\dot{8}}\alpha\right) \\ &\dot{>} \dot{1} \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{+} \left(\frac{\dot{1}}{\dot{4}}\alpha \dot{+} \frac{\dot{1}}{\dot{4}}\alpha\right) \dot{+} \left(\frac{\dot{1}}{\dot{8}}\alpha \dot{+} \frac{\dot{1}}{\dot{8}}\alpha \dot{+} \frac{\dot{1}}{\dot{8}}\alpha \dot{+} \frac{\dot{1}}{\dot{8}}\alpha\right) \\ &= \dot{1} \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{+} \frac{\dot{1}}{\dot{2}}\alpha = \dot{1} \dot{+} \frac{\dot{3}}{\dot{2}}\alpha \end{aligned}$$

$$S_{2^k} \dot{\geq} \dot{1} \dot{+} \frac{k}{\dot{2}}\alpha.$$

According to this, a sequence $\{S_{2^k}\}$ of $\{S_k\}$ is unbounded and thus the sequence $\{S_k\}$ is divergent and hence given series is divergent.

Theorem 1. *Let* ${}_\alpha\sum_{n=1}^{\infty} a_n$ *and* ${}_\alpha\sum_{n=1}^{\infty} b_n$ *be two non-Newtonian series with* ${}_\alpha\sum_{n=1}^{\infty} a_n = A$ *and* ${}_\alpha\sum_{n=1}^{\infty} b_n = B$*. Let* $\lambda \in \mathbb{R}(N)_\alpha$ *be also given. In this case, the following statements are true:*

a) The series ${}_\alpha\sum_{n=1}^{\infty}\left(a_n \dot{+} b_n\right)$ is convergent and ${}_\alpha\sum_{n=1}^{\infty}\left(a_n \dot{+} b_n\right) = A \dot{+} B$.

b) The series ${}_\alpha\sum_{n=1}^{\infty}\left(\lambda \dot{\times} a_n\right)$ is convergent and ${}_\alpha\sum_{n=1}^{\infty}\left(\lambda \dot{\times} a_n\right) = \lambda \dot{\times} A$.

Proof. a) Let ${}_\alpha\sum_{n=1}^{m} a_n = A_m$, ${}_\alpha\sum_{n=1}^{m} b_n = B_m$ and $S_m ={}_\alpha \sum_{n=1}^{m}\left(a_n \dot{+} b_n\right)$. Then $A_m \dot{+} B_m$. Hence

$${}^\alpha\lim_{m\to\infty} S_m = {}^\alpha\lim_{m\to\infty}\left(A_m \dot{+} B_m\right) = {}^\alpha\lim_{m\to\infty} A_m \dot{+} {}^\alpha\lim_{m\to\infty} B_m = A \dot{+} B,$$

thus the series ${}_\alpha\sum_{n=1}^{\infty}\left(a_n \dot{+} b_n\right)$ is convergent and ${}_\alpha\sum_{n=1}^{\infty}\left(a_n \dot{+} b_n\right) = A \dot{+} B$.

b) The proof is similar to the previous alternative. □

Theorem 2. (*Cauchy criterion*). *The series* ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ *is convergent iff when any number* $\varepsilon \dot{>} \dot{0}$ *is given, there exists at least a number* $m_0 \in \mathbb{N}$ *such that*

$$\left|S_{m+p} \dot{-} S_m\right| =_{\alpha} \sum_{n=m+1}^{m+p} S_n \dot{<} \varepsilon$$

for all $m > m_0$ *and all* $p \in \mathbb{N}$.

Proof. Let the series ${}_{\alpha}\sum_{n=1}^{m} a_n$ be convergent. Then, the sequence of partial sums $\{S_m\}$ of this series is also convergent. In this case, $\{S_m\}$ is a Caucy sequence. According to this, when any number $\varepsilon \dot{>} \dot{0}$ is given, there exists at least a number $m_0 \in \mathbb{N}$ such that

$$\left|S_{m+p} \dot{-} S_m\right| =_{\alpha} \sum_{n=m+1}^{m+p} S_n \dot{<} \varepsilon.$$

Conversely, when any number $\varepsilon \dot{>} \dot{0}$ is given, let there be a number $m_0 \in \mathbb{N}$ such that

$${}_{\alpha}\sum_{n=m+1}^{m+p} S_n \dot{<} \varepsilon$$

for all $m > m_0$ and all $p \in \mathbb{N}$. Then, the inequality $\left|S_{m+p} \dot{-} S_m\right| \dot{<} \varepsilon$ holds or all $m > m_0$ and all $p \in \mathbb{N}$. This shows that $\{S_m\}$ is a non-Newtonian Cauchy sequence. Since $\mathbb{R}(N)_{\alpha}$ is a non-Newtonian complete space, $\{S_m\}$ converges to an element of this space. Thus, the series ${}_{\alpha}\sum_{n=1}^{m} a_n$ is convergent. □

Theorem 3. *If the series* ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ *is convergent, then* ${}^{\alpha}\lim_{n\to\infty} a_n = \dot{0}$.

Proof. If it is taken $p = 1$ in the Cauchy criterion, then , there exists at least a number $m_0 \in \mathbb{N}$ such that

$$\left|{}_{\alpha}\sum_{n=m+1}^{m+p} a_n\right|_{\alpha} = \left|{}_{\alpha}\sum_{n=m+1}^{m+1} a_n\right|_{\alpha} = |a_{n+1}|_{\alpha} = \left|a_{n+1} \dot{-} \dot{0}\right|_{\alpha} \dot{<} \varepsilon$$

for all $m > m_0$, whenever any number $\varepsilon \dot{>} \dot{0}$ is given. This completes the proof. □

Example 3. *Since the series* ${}_{\alpha}\sum_{n=1}^{\infty} \frac{\dot{1}}{\dot{2}^{(n-1)_{\alpha}}}\alpha$ *is convergent,* ${}_{\alpha}\lim_{n\to\infty} \frac{\dot{1}}{\dot{2}^{(n-1)_{\alpha}}} = \dot{0}$.

Remark 1. *The opposite of the last theorem cannot be true. For example, although* $\frac{\dot{1}}{\dot{n}}\alpha \overset{\alpha}{\to} \dot{0}$ *as* $n \to \infty$, *we know that the series* ${}_{\alpha}\sum_{n=1}^{\infty} \frac{\dot{1}}{\dot{n}}\alpha$ *is divergent.*

Theorem 4. (*Comparison test*). *Suppose* $|b_n|_{\alpha} \dot{\leq} a_n$ *for all* $n \in \mathbb{N}$. *If the series* ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ *is convergent. then the series* ${}_{\alpha}\sum_{n=1}^{\infty} b_n$ *is also convergent.*

Proof. Since the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is convergent, it provides the Cauchy criterion. Thus, when any $\varepsilon \dot{>} \dot{0}$ is given, there is one $m_0 \in \mathbb{N}$ such that

$$\left|{}_{\alpha}\sum_{n=m+1}^{m+p} a_n\right|_{\alpha} = {}_{\alpha}\sum_{n=m+1}^{m+p} a_n \dot{<} \varepsilon$$

for all $m > m_0$ and $p \in \mathbb{N}$. Then, according to non-Newtonian triangle inequality and hypothesis,

$$\left|{}_{\alpha}\sum_{n=m+1}^{m+p} b_n\right|_{\alpha} ={}_{\alpha}\sum_{n=m+1}^{m+p} |b_n|_{\alpha} \dot{\leq} {}_{\alpha}\sum_{n=m+1}^{m+p} a_n \dot{<} \varepsilon$$

for all $m > m_0$ and $p \in \mathbb{N}$. In this case, the series ${}_{\alpha}\sum_{n=1}^{\infty} b_n$ is convergent by Cauchy criteria. □

Definition 8. *If the series ${}_{\alpha}\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent(α-absolutely convergent). However, if the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is convergent but the series ${}_{\alpha}\sum_{n=1}^{\infty} |a_n|$ is divergent, then the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is said to be conditional convergent(α- conditional convergent).*

Theorem 5. *If the series ${}_{\alpha}\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is also convergent.*

Proof. Since the series ${}_{\alpha}\sum_{n=1}^{\infty} |a_n|$ is convergent, it provides the Cauchy criterion. Thus, when any $\varepsilon \dot{>} \dot{0}$ is given, there is one $m_0 \in \mathbb{N}$ such that

$$\left|{}_{\alpha}\sum_{n=m+1}^{m+p} |a_n|_{\alpha}\right|_{\alpha} = {}_{\alpha}\sum_{n=m+1}^{m+p} |a_n|_{\alpha} \dot{<} \varepsilon$$

for all $m > m_0$ and $p \in \mathbb{N}$. Then, according to non-Newtonian triangle inequality and hypothesis,

$$\left|{}_{\alpha}\sum_{n=m+1}^{m+p} a_n\right|_{\alpha} \dot{\leq} {}_{\alpha}\sum_{n=m+1}^{m+p} |a_n|_{\alpha} = \left|{}_{\alpha}\sum_{n=m+1}^{m+p} |a_n|_{\alpha}\right|_{\alpha} \dot{<} \varepsilon$$

for all $m > m_0$ and $p \in \mathbb{N}$. This completes the proof. □

Theorem 6. *(Cauchy condensation test). Let $\{a_n\}$ be an α-monotone decreasing sequence with positive terms. Then, the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is convergent iff the series ${}_{\alpha}\sum_{n=1}^{\infty} \dot{2}^{k_{\alpha}} \dot{\times} a_{2^k}$ is convergent.*

Proof. Suppose that the series ${}_{\alpha}\sum_{n=1}^{\infty} a_n$ is convergent. Using that the sequence $\{a_n\}$ is α-monotone decreasing, we have

$$\begin{aligned}\dot{2}^{(k-1)_{\alpha}} \dot{\times} a_{2^k} &= \alpha\left(2^{k-1}.\alpha^{-1}(a_{2^k})\right) = a_{2^k} \dot{+} a_{2^k} \dot{+} ... \dot{+} a_{2^k} \\ &\dot{\leq} a_{2^{k-1}+1} \dot{+} a_{2^{k-1}+2} \dot{+} ... \dot{+} a_{2^k} \\ &= {}_{\alpha}\sum_{m=2^{k-1}+1}^{2^k} a_m\end{aligned}$$

and thus

$${}_{\alpha}\sum_{k=1}^{n} \dot{2}^{(k-1)_{\alpha}} \dot{\times} a_{2^k} \dot{\leq} {}_{\alpha}\sum_{k=1}^{n}\left({}_{\alpha}\sum_{m=2^{k-1}+1}^{2^k} a_m\right) = {}_{\alpha}\sum_{m=2}^{2^n} a_m.$$

Since the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\left\{{}_\alpha\sum_{m=2}^{2^n} a_m\right\}$ is convergent and so bounded. Then, the sequence $\{A_n\}$ with the general term $A_n = {}_\alpha\sum_{k=1}^{n} \dot{2}^{(k-1)_\alpha} \dot{\times} a_{2^k}$ is also bounded. At the same time, the sequence $\{A_n\}$ is α-monotonically increasing, since the sequence $\{a_n\}$ has the positive terms. Hence the sequence $\{A_n\}$ is convergent and the partial sums sequence $\{S_n\}$ with $S_n = {}_\alpha\sum_{k=1}^{n} \dot{2}^{k_\alpha} \dot{\times} a_{2^k}$ is convergent. This means that the series ${}_\alpha\sum_{k=1}^{\infty} \dot{2}^{k_\alpha} \dot{\times} a_{2^k}$ is convergent.

Conversely, suppose that the series ${}_\alpha\sum_{k=1}^{\infty} \dot{2}^{k_\alpha} \dot{\times} a_{2^k}$ is convergent. By the inequality

$$\begin{aligned}
{}_\alpha\sum_{m=2^{k-1}+1}^{2^k} a_m &= a_{2^{k-1}+1} \dot{+} a_{2^{k-1}+2} \dot{+} \ldots \dot{+} a_{2^{k-1}+2^{k-1}-1} \dot{+} a_{2^k} \\
&\dot{\leq} a_{2^{k-1}} \dot{+} a_{2^{k-1}} \dot{+} \ldots \dot{+} a_{2^{k-1}} \\
&= \dot{2}^{(k-1)_\alpha} \dot{\times} a_{2^{k-1}},
\end{aligned}$$

we have the inequality

$${}_\alpha\sum_{m=2}^{2^n} a_m = {}_\alpha\sum_{k=1}^{n}\left({}_\alpha\sum_{m=2^{k-1}+1}^{2^k} a_m\right) \dot{\leq} {}_\alpha\sum_{k=1}^{n} \dot{2}^{(k-1)_\alpha} \dot{\times} a_{2^k}.$$

By the hypothesis, the sequence $\left\{{}_\alpha\sum_{k=1}^{n} \dot{2}^{(k-1)_\alpha} \dot{\times} a_{2^k}\right\}$ is convergent and thus bounded. Hence, the sequence $\{B_n\}$ with $B_n = {}_\alpha\sum_{m=2}^{2^n} a_m$ is also bounded. At the same time, since the sequence $\{a_n\}$ has the positive terms, the sequence $\{B_n\}$ is convergent and consequently the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is convergent. □

Example 4. *(non-Newtonian geometri series). We investigate the convergence condition of the series* ${}_\alpha\sum_{k=1}^{\infty} a \dot{\times} r^{(k-1)_\alpha}$ *with* $r \neq \dot{0}$*:*

Let $\{S_n\}$ be the partial sums sequence of the series ${}_\alpha\sum_{k=1}^{\infty} a \dot{\times} r^{(k-1)_\alpha}$. Clearly, this series is convergent for $a = \dot{0}$. We take $a \neq \dot{0}$. We have $S_n = n \dot{\times} a$, if $r = \dot{1}$. Suppose now that the sequence $\{S_n\}$ is bounded. Then, there is one $M \dot{>} \dot{0}$ with $|S_n|_\alpha \dot{\leq} M$ for all $n \in \mathbb{N}$. In this case, $\left|\alpha^{-1}(S_n)\right| = \left|n.\alpha^{-1}(a)\right| \leq \alpha^{-1}(M)$ and hence the real number sequence $\left\{\left|\alpha^{-1}(S_n)\right|\right\}$ is bounded. This creates a contradiction, since $\alpha^{-1}(a) \neq 0$ for $a \neq \dot{0}$. According to this, the sequence $\{S_n\}$ for $r = \dot{1}$ is unbounded and thus divergent. We have

$$\begin{aligned}
S_n &= {}_\alpha\sum_{k=1}^{n} a \dot{\times} r^{(k-1)_\alpha} = a \dot{+} a \dot{\times} r \dot{+} a \dot{\times} r^{2_\alpha} \dot{+} \ldots \dot{+} a \dot{\times} r^{(n-1)_\alpha} \\
&= \alpha\left[\alpha^{-1}(a) + \alpha^{-1}(a).\alpha^{-1}(r) + \alpha^{-1}(a).\alpha^{-1}(r)^2 + \ldots + \alpha^{-1}(a).\alpha^{-1}(r)^{n-1}\right] \\
&= \alpha\left[\alpha^{-1}(a).\frac{1-\alpha^{-1}(r)^n}{1-\alpha^{-1}(r)}\right] = a \dot{\times} \frac{\dot{1} \dot{-} r^{n_\alpha}}{\dot{1} \dot{-} r}\alpha
\end{aligned}$$

for all $n \in \mathbb{N}$ and using this

$$^{\alpha}\lim_{n\to\infty} S_n = {}^{\alpha}\lim_{n\to\infty}\left(a \dot{\times} \frac{1 \dot{-} r^{n_\alpha}}{1 \dot{-} r}\alpha\right) = \frac{a}{1 \dot{-} r}\alpha \dot{-} \frac{a}{1 \dot{-} r}\alpha \dot{\times} {}^{\alpha}\lim_{n\to\infty} r^{n_\alpha}.$$

If $|r|_\alpha \dot{<} \dot{1}$, then $r^{n_\alpha} \xrightarrow{\alpha} \dot{0}$ as $n \to \infty$ and thus the series ${}_\alpha\sum_{k=1}^{\infty} a \dot{\times} r^{(k-1)_\alpha}$ is convergent, and also ${}_\alpha\sum_{k=1}^{\infty} a \dot{\times} r^{(k-1)_\alpha} = \frac{a}{\dot{1}\dot{-}r}\alpha$. If $|r|_\alpha \dot{>} \dot{1}$, then the series ${}_\alpha\sum_{k=1}^{\infty} a \dot{\times} r^{(k-1)_\alpha}$ is divergent, since the sequence $\{r^{n_\alpha}\}$ is divergent.

Corollary 1. *(p-test) The series ${}_\alpha\sum_{n=1}^{\infty} \frac{\dot{1}}{\dot{n}^{p_\alpha}}\alpha$ is convergent for $p > 1$ and is divergent for $p \leq 1$.*

Proof. Since $\frac{\dot{1}}{\dot{n}^{p_\alpha}}\alpha \to \dot{+}\infty \neq \dot{0}$ for $p < 0$ and $\frac{\dot{1}}{\dot{n}^{p_\alpha}}\alpha = \dot{1} \neq \dot{0}$ for $p = 0$, the series $\sum_{n=1}^{\infty} \frac{\dot{1}}{\dot{n}^{p_\alpha}}\alpha$ is divergent for $p \leq 1$. Let $p > 0$ be now accepted. Then, we have the geometric series

$${}_\alpha\sum_{n=1}^{\infty} \dot{2}^{n_\alpha} \dot{\times} \frac{\dot{1}}{\left(\dot{2}^{n_\alpha}\right)^{p_\alpha}}\alpha = {}_\alpha\sum_{n=1}^{\infty} \dot{2}^{(1-p)_\alpha}.$$

This series is divergent for $0 < p \leq 1$ and the series $\sum_{n=1}^{\infty} \frac{\dot{1}}{\dot{n}^{p_\alpha}}\alpha$ is divergent, according to Cauchy condensation test. The series ${}_\alpha\sum_{n=1}^{\infty} \dot{2}^{n_\alpha} \dot{\times} \frac{\dot{1}}{\left(\dot{2}^{n_\alpha}\right)^{p_\alpha}}\alpha$ is convergent for $p > 1$ and and the series ${}_\alpha\sum_{n=1}^{\infty} \frac{\dot{1}}{\dot{n}^{p_\alpha}}\alpha$ is convergent, according to Cauchy condensation test. □

Theorem 7. *(Cauchy's root test). Let a series ${}_\alpha\sum_{n=1}^{\infty} a_n$ be given and be $L = {}^{\alpha}\limsup |a_n|_\alpha^{(1/n)_\alpha}$. Then the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is*

- (i) absolutely convergent, if $L \dot{<} \dot{1}$
- (ii) divergent, if $L \dot{>} \dot{1}$
- (iii) not said to be convergent or divergent, if $L = \dot{1}$.

Proof. (i) There is at least $\varepsilon \dot{>} \dot{0}$ such that $L \dot{+} \varepsilon \dot{<} \dot{1}$, if $L \dot{<} \dot{1}$. By the definition ${}^{\alpha}\limsup$, there is one $n_0 \in \mathbb{N}$ such that

$$\left|{}^{\alpha}\sup_{n\geq k} |a_n|_\alpha^{(1/n)_\alpha} \dot{-} L\right|_\alpha \dot{<} \varepsilon$$

and hence

$$L \dot{-} \varepsilon \dot{<} {}^{\alpha}\sup_{n\geq k} |a_n|_\alpha^{(1/n)_\alpha} < L \dot{+} \varepsilon$$

for all $k > n_0$. In particular, we have

$$|a_n|_\alpha^{(1/n)_\alpha} \dot{<} L \dot{+} \varepsilon$$

and thus

$$||a_n|_\alpha|_\alpha = |a_n|_\alpha \dot{<} \left(L \dot{+} \varepsilon\right)^{n_\alpha}$$

for all $n > n_0$. Since the geometric series ${}_\alpha\sum_{n=n_0+1}^{\infty}\left(L\dot{+}\varepsilon\right)^{n_\alpha}$ with $\dot{0}\dot{<}L\dot{+}\varepsilon\dot{<}\dot{1}$ is convergent, the series ${}_\alpha\sum_{n=n_0+1}^{\infty}|a_n|_\alpha$ is convergent by the comparison test.

(ii) Let $L\dot{>}\dot{1}$. By the definition α-upper limit, there exists a subsequence, converging to L, of the sequence $\left\{|a_n|_\alpha^{(1/n)_\alpha}\right\}$. This means that $|a_n|_\alpha\dot{>}\dot{1}$ for infinite selection of n. Then ${}^\alpha\lim_{n\to\infty} a_n=\dot{0}$ cannot hold. Thus, the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) $L=\dot{1}$ for two series ${}_\alpha\sum_{n=1}^{\infty}\frac{\dot{1}}{n}\alpha$ and ${}_\alpha\sum_{n=1}^{\infty}\frac{\dot{1}}{n^{2_\alpha}}\alpha$. But this two series diverges and converges, respectively. □

Theorem 8. *(Cauchy's rate test). Let a series ${}_\alpha\sum_{n=1}^{\infty} a_n$ be given. Let $a_n\neq\dot{0}$ for all $n\in\mathbb{N}$ and $L={}^\alpha\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\alpha\right|_\alpha$. Then*

(i) the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is absolutely convergent, if $L\dot{<}\dot{1}$

(ii) the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is divergent, if $L\dot{>}\dot{1}$

(iii) This test does not give, $L=\dot{1}$.

Proof. (i) If $L\dot{<}\dot{1}$, then we can choose one $\varepsilon\dot{>}\dot{0}$ such that $\dot{0}\dot{<}L\dot{+}\varepsilon\dot{<}\dot{1}$. By the hypothesis, there is one $n_0\in\mathbb{N}$ such that

$$\left|\left|\frac{a_{n+1}}{a_n}\alpha\right|_\alpha\dot{-}L\right|_\alpha\dot{<}\varepsilon$$

and thus

$$\left|\frac{a_{n+1}}{a_n}\alpha\right|_\alpha\dot{<}\varepsilon\dot{+}L\dot{<}\dot{1}$$

for all $n\dot{>}n_0$. If we say $S=\varepsilon\dot{+}L$, then we can write

$$|a_{n+1}|_\alpha\dot{<}|a_n|_\alpha\dot{\times}S$$
$$|a_{n+2}|_\alpha\dot{<}|a_{n+1}|_\alpha\dot{\times}S\dot{<}|a_n|_\alpha\dot{\times}S^{2_\alpha}$$

for all $n\dot{>}n_0$. Thus, we have

$$||a_n|_\alpha|_\alpha=|a_n|_\alpha\dot{\leq}|a_{n_0+1}|_\alpha\dot{\times}S^{(n-n_0-1)_\alpha}.$$

If we use $S\dot{<}\dot{1}$, then the non-Newtonian geometric series ${}_\alpha\sum_{n=n_0+1}^{\infty}|a_{n_0+1}|_\alpha\dot{\times}S^{(n-n_0-1)_\alpha}$ is convergent. Then, by the comparison test, the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $L\dot{>}\dot{1}$, then we can choose one $\varepsilon\dot{>}\dot{0}$ such that $L\dot{-}\varepsilon\dot{>}\dot{1}$. By the hypothesis, there is one $n_0\in\mathbb{N}$ such that

$$\left|\left|\frac{a_{n+1}}{a_n}\alpha\right|_\alpha\dot{-}L\right|_\alpha\dot{<}\varepsilon$$

and thus

$$\left|\frac{a_{n+1}}{a_n}\alpha\right|_\alpha\dot{>}L\dot{-}\varepsilon\dot{>}\dot{1}$$

for all $n\dot{>}n_0$. If we say $S=L\dot{-}\varepsilon$, then we can write then we can write

$$|a_{n+1}|_\alpha\dot{>}|a_n|_\alpha\dot{\times}S$$
$$|a_{n+2}|_\alpha\dot{>}|a_{n+1}|_\alpha\dot{\times}S\dot{>}|a_n|_\alpha\dot{\times}S^{2_\alpha}$$

for all $n \dot{>} n_0$. Thus, we have

$$|a_n|_\alpha \dot{\geq} |a_{n_0+1}|_\alpha \dot{\times} S^{(n-n_0-1)_\alpha}.$$

If we use $S \dot{>} \dot{1}$, then the non-Newtonian geometric series ${}_\alpha\sum_{n=n_0+1}^{\infty} |a_{n_0+1}|_\alpha \dot{\times} S^{(n-n_0-1)_\alpha}$ is divergent. Then, by the comparison test, the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) The proof is as in Cauchy's root test. □

Theorem 9. *(Leibnitz's Test). If $a_1 \dot{\geq} a_2 \dot{\geq} ... \dot{\geq} a_n \dot{\geq} ... \dot{\geq} \dot{0}$ and ${}^\alpha\lim_{n\to\infty} a_n = \dot{0}$, then the non-Newtonian α-alterne series ${}_\alpha\sum_{n=1}^{\infty} \left(\dot{0} \dot{-} \dot{1}\right)^{(n-1)_\alpha}$ is convergent.*

Proof. Let the partial sums sequence of the series ${}_\alpha\sum_{n=1}^{\infty} \left(\dot{0} \dot{-} \dot{1}\right)^{(n-1)_\alpha}$ be (S_n). Then, we have

$$\begin{aligned} S_{2n} &= a_1 \dot{+} \left(\dot{0} \dot{-} a_2\right) \dot{+} ... \dot{+} a_{2n-1} \dot{+} \left(\dot{0} \dot{-} a_{2n}\right) \\ &= \left(a_1 \dot{-} a_2\right) \dot{+} ... \dot{+} \left(a_{2n-1} \dot{-} a_{2n}\right) \\ &\dot{\leq} \left(a_1 \dot{-} a_2\right) \dot{+} ... \dot{+} \left(a_{2n-1} \dot{-} a_{2n}\right) \dot{+} \left(a_{2n-1} \dot{-} a_{2n}\right) \\ &= S_{2n+2} \end{aligned}$$

and thus $S_2 \dot{\leq} S_4 \dot{\leq} ... \dot{\leq} S_{2n} \dot{\leq} ....$ Since $a_n \dot{-} a_{n+1} \dot{\geq} \dot{0}$ for all $n \in \mathbb{N}$ by the hypothesis,

$$S_{2n} = a_1 \dot{-} \left(a_2 \dot{-} a_3\right) \dot{-} \left(a_4 \dot{-} a_5\right) \dot{-} ... \dot{-} \left(a_{2n-2} \dot{-} a_{2n-1}\right) \dot{-} a_{2n} \dot{\leq} a_1$$

and thus, the sequence (S_{2n}) is α-bounded above and also α-convergent. Let us accept that ${}^\alpha\lim_{n\to\infty} S_{2n} = S$. Using the equalities $S_{2n-1} = S_{2n} \dot{+} a_{2n}$ and ${}^\alpha\lim_{n\to\infty} a_{2n} = \dot{0}$, we have

$${}^\alpha\lim_{n\to\infty} S_{2n-1} = {}^\alpha\lim_{n\to\infty} S_{2n-1} \dot{+} {}^\alpha\lim_{n\to\infty} a_{2n} = S.$$

Consequently, we have ${}^\alpha\lim_{n\to\infty} S_n = S$ from the equalities ${}^\alpha\lim_{n\to\infty} S_{2n} = S$ and ${}^\alpha\lim_{n\to\infty} S_{2n-1} = S$. This completes the proof. □

Theorem 10. *(Abel's partial α-sums formula). Let $\{\varphi_n\}$ and (γ_n) be two non-Newtonian real number sequences. Let be also $T_n = {}_\alpha\sum_{k=1}^{n} \gamma_k$ for all $n \in \mathbb{N}$ and $T_0 = \dot{0}$. Then, the inequality*

$${}_\alpha\sum_{k=1}^{n} \varphi_k \dot{\times} \gamma_k = \varphi_n \dot{\times} T_n \dot{-} {}_\alpha\sum_{k=1}^{n-1} \left(\varphi_{k+1} \dot{-} \varphi_k\right) \dot{\times} T_k$$

holds for all $n \in \mathbb{N}$.

Proof. We have

$$\begin{aligned}
{}_{\alpha}\sum_{k=1}^{n} \varphi_k \dot{\times} \gamma_k &= \varphi_1 \dot{\times} \gamma_1 \dot{+} \varphi_2 \dot{\times} \gamma_2 \dot{+} ... \dot{+} \varphi_n \dot{\times} \gamma_n \\
&= \varphi_1 \dot{\times} T_1 \dot{+} \varphi_2 \dot{\times} \left(T_2 \dot{-} T_1\right) \dot{+} ... \dot{+} \varphi_n \dot{\times} \left(T_n \dot{-} T_{n-1}\right) \\
&= {}_{\alpha}\sum_{k=1}^{n-1} \left(\varphi_k \dot{-} \varphi_{k+1}\right) \dot{\times} T_k \dot{+} \varphi_n \dot{\times} T_n \\
&= \varphi_n \dot{\times} T_n \dot{-} {}_{\alpha}\sum_{k=1}^{n-1} \left(\varphi_{k+1} \dot{-} \varphi_k\right) \dot{\times} T_k
\end{aligned}$$

and this completes the proof. □

Theorem 11. *(Abel's test). Let $\{a_n\}$ and (b_n) be two non-Newtonian real number sequences. If the conditions (i) the series ${}_{\alpha}\sum_{k=1}^{\infty} b_n$ is convergent and (ii) the sequence $\{a_n\}$ is α-monotone and α-bounded, holds, then the series ${}_{\alpha}\sum_{k=1}^{\infty} a_n \dot{\times} b_n$ is convergent.*

Proof. Since the sequence $\{a_n\}$ is α-bounded, there is one $\dot{K} \dot{>} \dot{0}$ such that $|a_n|_{\alpha} \dot{\leq} K$ for all $n \in \mathbb{N}$. By Cauchy's criterion, when any number $\varepsilon \dot{>} \dot{0}$ is given, there is at least $n_0 \in \mathbb{N}$ such that

$$\left| {}_{\alpha}\sum_{k=n+1}^{n+p} b_k \right|_{\alpha} = \left| {}_{\alpha}\sum_{k=1}^{p} b_{n+k} \right| \dot{<} \frac{\varepsilon}{\dot{3} \dot{\times} K}\alpha$$

for all $n > n_0$ and $p \in \mathbb{N}$. We also have

$${}_{\alpha}\sum_{k=n+1}^{n+p} a_k \dot{\times} b_k = {}_{\alpha}\sum_{k=1}^{p} a_{n+k} \dot{\times} b_{n+k}.$$

for all $n, p \in \mathbb{N}$. If we take $\varphi_k = a_{n+k}$ and $\gamma_k = b_{n+k}$ in Abel's partial sums formula and say $T_p = {}_{\alpha}\sum_{k=1}^{p} b_{n+k}$, then we have

$${}_{\alpha}\sum_{k=n+1}^{n+p} a_k \dot{\times} b_k = {}_{\alpha}\sum_{k=1}^{p} a_{n+k} \dot{\times} b_{n+k} = a_{n+p} \dot{\times} T_p \dot{-} {}_{\alpha}\sum_{k=1}^{p-1} \left(a_{n+k+1} \dot{-} a_{n+k}\right) \dot{\times} T_k.$$

If $\{a_n\}$ is an α-monotone increasing, then

$$\begin{aligned}
{}_{\alpha}\sum_{k=1}^{p-1} \left|a_{n+k+1} \dot{-} a_{n+k}\right|_{\alpha} &= {}_{\alpha}\sum_{k=1}^{p-1} \left(a_{n+k+1} \dot{-} a_{n+k}\right) \\
&= a_{n+2} \dot{-} a_{n+1} \dot{+} a_{n+3} \dot{-} a_{n+2} \dot{+} ... \dot{+} a_{n+p} \dot{-} a_{n+p-1} \\
&= a_{n+p} \dot{-} a_{n+1} = \left|a_{n+p} \dot{-} a_{n+1}\right|_{\alpha}.
\end{aligned}$$

Similarily, If $\{a_n\}$ is an α-monotone increasing, then

$${}_{\alpha}\sum_{k=1}^{p-1} \left|a_{n+k+1} \dot{-} a_{n+k}\right|_{\alpha} = {}_{\alpha}\sum_{k=1}^{p-1} \left(a_{n+k} \dot{-} a_{n+k+1}\right) = a_{n+1} \dot{-} a_{n+p} = \left|a_{n+p} \dot{-} a_{n+1}\right|_{\alpha}.$$

Thus, If $\{a_n\}$ is an α-monotone sequence, then

$$_\alpha\sum_{k=1}^{p-1}\left|a_{n+k+1}\dot{-}a_{n+k}\right|_\alpha = \left|a_{n+p}\dot{-}a_{n+1}\right|_\alpha .$$

Hence

$$\begin{aligned}\left|_\alpha\sum_{k=n+1}^{n+p} a_k\dot{\times}b_k\right|_\alpha &\dot{\leq} |a_{n+p}|_\alpha\dot{\times}|T_p|_\alpha\dot{+}\ _\alpha\sum_{k=1}^{p-1}\left|a_{n+k+1}\dot{-}a_{n+k}\right|_\alpha\dot{\times}|T_k|\\ &\dot{<}\frac{\varepsilon}{\dot{3}\dot{\times}K}\alpha\dot{\times}\left(|a_{n+p}|_\alpha\dot{+}_\alpha\sum_{k=1}^{p-1}\left|a_{n+k+1}\dot{-}a_{n+k}\right|_\alpha\right)\\ =&\ \frac{\varepsilon}{\dot{3}\dot{\times}K}\alpha\dot{\times}\left(|a_{n+p}|_\alpha\dot{+}\left|a_{n+p}\dot{-}a_{n+1}\right|_\alpha\right)\\ =&\ \frac{\varepsilon}{\dot{3}\dot{\times}K}\alpha\dot{\times}\left(\dot{2}\dot{\times}|a_{n+p}|_\alpha\dot{+}|a_{n+1}|_\alpha\right)\dot{\leq}\frac{\varepsilon}{\dot{3}\dot{\times}K}\alpha\dot{\times}\left(\dot{3}\dot{\times}K\right)=\varepsilon\end{aligned}$$

and thus the series $_\alpha\sum_{k=1}^{n} a_n\dot{\times}b_n$ is convergent by Cauchy criterion. □

Theorem 12. *(Dirichlet's test). Let $\{a_n\}$ and (b_n) be two non-Newtonian real number sequences. If the conditions (i) The partial sums sequence of the series $_\alpha\sum_{k=1}^{\infty} b_n$ is bounded and (ii) the sequence $\{a_n\}$ is α-monotone and $^\alpha\lim_{n\to\infty} a_n=\dot{0}$, holds, then the series $_\alpha\sum_{k=1}^{\infty} a_n\dot{\times}b_n$ is convergent.*

Proof. Let the α-partial sum of the series $_\alpha\sum_{k=1}^{\infty} b_n$ be t_n, i. e. $t_n={}_\alpha\sum_{k=1}^{n} b_k$. By the alternative (i), there is one $M\dot{>}\dot{0}$ such that $|t_n|_\alpha\dot{<}M$ for all $n\in\mathbb{N}$. By the alternative (ii) again, given any $\varepsilon\dot{>}\dot{0}$, there is one $n_0\in\mathbb{N}$ such that $|a_n|_\alpha\dot{<}\frac{\varepsilon}{\dot{6}\dot{\times}M}\alpha$ for all $n>n_0$. Then, taken $\varphi_k=a_{n+k}$, $\gamma_k=b_{n+k}$ and $T_p={}_\alpha\sum_{k=1}^{p} b_{n+k}$ for all $p\in\mathbb{N}$ in Abel's α-sums formula, we write

$$_\alpha\sum_{k=n+1}^{n+p} a_k\dot{\times}b_k={}_\alpha\sum_{k=1}^{p} a_{n+k}\dot{\times}b_{n+k}=a_{n+p}\dot{\times}T_p\dot{-}\ _\alpha\sum_{k=1}^{p-1}\left(a_{n+k+1}\dot{-}a_{n+k}\right)\dot{\times}T_k.$$

Since the sequence $\{a_n\}$ is monotone, we have

$$\begin{aligned}\left|_\alpha\sum_{k=n+1}^{n+p} a_k\dot{\times}b_k\right|_\alpha &\dot{\leq}|a_{n+p}|_\alpha\dot{\times}|T_p|_\alpha\dot{+}\ _\alpha\sum_{k=1}^{p-1}\left|a_{n+k+1}\dot{-}a_{n+k}\right|_\alpha\dot{\times}|T_k|_\alpha\\ =&\ |a_{n+p}|_\alpha\dot{\times}\left|t_{n+p}\dot{-}t_n\right|_\alpha\dot{+}\ _\alpha\sum_{k=1}^{p-1}\left|a_{n+k+1}\dot{-}a_{n+k}\right|_\alpha\dot{\times}\left|t_{n+k}\dot{-}t_n\right|_\alpha\\ &\dot{\leq}|a_{n+p}|_\alpha\dot{\times}(|t_{n+p}|_\alpha\dot{+}|t_n|_\alpha)\dot{+}\ _\alpha\sum_{k=1}^{p-1}\left|a_{n+k+1}\dot{-}a_{n+k}\right|_\alpha\dot{\times}(|t_{n+}|_\alpha\dot{+}|t_n|_\alpha)\\ &\dot{\leq}|a_{n+p}|_\alpha\dot{\times}\left(\dot{2}\dot{\times}M\right)\dot{+}\ _\alpha\sum_{k=1}^{p-1}\left|a_{n+k+1}\dot{-}a_{n+k}\right|_\alpha\dot{\times}\left(\dot{2}\dot{\times}M\right)\\ =&\ \left(\dot{2}\dot{\times}M\right)\dot{\times}\left(|a_{n+p}|_\alpha\dot{+}\ _\alpha\sum_{k=1}^{p-1}\left|a_{n+k+1}\dot{-}a_{n+k}\right|_\alpha\right)\\ =&\ \left(\dot{2}\dot{\times}M\right)\dot{\times}\left(|a_{n+p}|_\alpha\dot{+}\left|a_{n+p}\dot{-}a_{n+1}\right|_\alpha\right)\\ &\dot{\leq}\left(\dot{2}\dot{\times}M\right)\dot{\times}\left(|a_{n+p}|_\alpha\dot{+}|a_{n+p}|_\alpha\dot{+}|a_{n+1}|_\alpha\right)\\ &\dot{<}\left(\dot{2}\dot{\times}M\right)\dot{\times}\left(\frac{\varepsilon}{\dot{6}\dot{\times}M}\dot{+}\frac{\varepsilon}{\dot{6}\dot{\times}M}\dot{+}\frac{\varepsilon}{\dot{6}\dot{\times}M}\right)\\ =&\ \varepsilon\end{aligned}$$

for all $n>n_0$ and $p\in\mathbb{N}$. Then, by the Cauchy criterion, the series $_\alpha\sum_{k=1}^{\infty} a_n\dot{\times}b_n$ is convergent.

□

Theorem 13. *Let the series* ${}_\alpha\sum_{k=1}^{\infty} a_n$ *be convergent. Let us agree that* $a_n^+ = \frac{|a_n|_\alpha \dot{+} a_n}{\dot{2}}\alpha$ *and* $a_n^- = \frac{|a_n|_\alpha \dot{-} a_n}{\dot{2}}\alpha$ *for all* $n \in \mathbb{N}$.

(i) If ${}_\alpha\sum_{k=1}^{\infty} a_n$ *is conditionally convergent, then both* ${}_\alpha\sum_{k=1}^{\infty} a_n^+$ *and* ${}_\alpha\sum_{k=1}^{\infty} a_n^-$ *are divergent. If* ${}_\alpha\sum_{k=1}^{\infty} a_n$ *is divergent, then both* ${}_\alpha\sum_{k=1}^{\infty} a_n^+$ *and* ${}_\alpha\sum_{k=1}^{\infty} a_n^-$ *are divergent.*

(ii) If ${}_\alpha\sum_{k=1}^{\infty} a_n$ *is absolutely convergent, then both* ${}_\alpha\sum_{k=1}^{\infty} a_n^+$ *and* ${}_\alpha\sum_{k=1}^{\infty} a_n^-$ *are convergent, furthermore the following equation holds:*

$${}_\alpha\sum_{k=1}^{\infty} a_n = {}_\alpha\sum_{k=1}^{\infty} a_n^+ \dot{-} {}_\alpha\sum_{k=1}^{\infty} a_n^-.$$

Proof. $a_n^+ = a_n$ and $a_n^- = \dot{0}$ if $a_n \dot{\geq} \dot{0}$, $a_n^+ = \dot{0}$ and $a_n^- = \dot{0} \dot{-} a_n$ if $a_n \dot{\leq} \dot{0}$. Also $a_n = a_n^+ \dot{-} a_n^-$ and $|a_n|_\alpha = a_n^+ \dot{+} a_n^-$ for all $n \in \mathbb{N}$.

(i) Let the series ${}_\alpha\sum_{k=1}^{\infty} a_n$ be conditionally convergent. *So*, the series ${}_\alpha\sum_{k=1}^{\infty} a_n$ is convergent and the series ${}_\alpha\sum_{k=1}^{\infty} |a_n|_\alpha$ is divergent. Let us assume that one of the series ${}_\alpha\sum_{k=1}^{\infty} a_n^+$ and ${}_\alpha\sum_{k=1}^{\infty} a_n^-$ is convergent. Since $a_n = a_n^+ \dot{-} a_n^-$ for all $n \in \mathbb{N}$, both of the series ${}_\alpha\sum_{k=1}^{\infty} a_n^+$ and ${}_\alpha\sum_{k=1}^{\infty} a_n^-$ are convergent. Similarly, since $a_n = a_n^+ \dot{-} a_n^-$ for all $n \in \mathbb{N}$, if both of the series ${}_\alpha\sum_{k=1}^{\infty} a_n^+$ and ${}_\alpha\sum_{k=1}^{\infty} a_n^-$ are convergent, then ${}_\alpha\sum_{k=1}^{\infty} a_n$ is also convergent. This creates a contradiction and completes the proof.

(ii) Let the series ${}_\alpha\sum_{k=1}^{\infty} |a_n|_\alpha$ be c.onditionally convergent. Then the series ${}_\alpha\sum_{k=1}^{\infty} a_n$ is convergent and both of the series

$${}_\alpha\sum_{k=1}^{\infty} a_n^+ = {}_\alpha\sum_{k=1}^{\infty}\left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \left(|a_n|_\alpha \dot{+} a_n\right)\right) \text{ and } {}_\alpha\sum_{k=1}^{\infty} a_n^- = {}_\alpha\sum_{k=1}^{\infty}\left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \left(|a_n|_\alpha \dot{-} a_n\right)\right)$$

are convergent. Furthermore, since $a_n = a_n^+ \dot{-} a_n^-$ for all $n \in \mathbb{N}$, the following equality:

$${}_\alpha\sum_{k=1}^{\infty} a_n = {}_\alpha\sum_{k=1}^{\infty} a_n^+ \dot{-} {}_\alpha\sum_{k=1}^{\infty} a_n^-.$$

□

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