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Two Variable Cubic Spline Interpolation

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Abstract: *Two variable natural cubic spline interpolation formula is derived and illustrated using an example*
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I. Introduction

If the explicit nature of the function $y = f(x)$ is not known but the set of tabular values satisfying the function is known then the process of replacing $f(x)$ by suitable function say g(x) satisfying the given set of tabular values is called interpolation. If g(x) is a polynomial then it is called polynomial interpolation [4]. In many cases it is seen that polynomial oscillates varyingly but the function varies smoothly [1]. To overcome this spline function is considered which is a function of polynomial bits joined together. The cubic spline procedure has sufficient flexibility due to the four constantsinvolved in a general cubic polynomial which ensures the condition that the interpolant is continuously differentiable in the interval and has continuous second derivative [3]. Thus because of their smoothness conditions the most frequently used spline interpolation is the cubic spline interpolation. A two variable cubic spline interpolation of a function $z = f(x, y)$ is the fitting of a unique series of cubic splines for a given set of data points (x_i, y_j, z_{ij}). The points (x, y) at which f(x, y) are known lie on a grid in the x-y plane. In order to derive a two variable natural cubic spline the existence of continuity condition of the spline function and its partial derivatives at the edge of each grid are assumed.

II. Two Variable Natural Cubic Spline

Consider the division rectangle $I = [a,b] \times [c,d]$. Let $a = x_1 < x_2 < ---- < x_n = b$ and $c = y_1 < y_2 < ---- < y_m = d$ be the set of data points satisfied by $f(x_i, y_j) = z_{ij}$ for all i = 1, 2 ,…., n and j = 1,2,…..,m.[2] . A two variable cubic spline $S_{ij}(x, y)$ is a unique function coinciding with z_{ij} in each rectangular grid $I_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for all i = 1, 2 ,….,(n-1) and j = 1,2,…..,(m-1). Since $S_{ij}(x, y)$ is a cubic spline in two variables its all second order partial derivatives should be linear and continuous. Here we are considering the second order partial derivative with respect to x

Let $$\frac{\partial^2 S_{ij}}{\partial x^2} = \frac{M_i(x_{i+1} - x)}{h_i} + \frac{M_{i+1}(x - x_i)}{h_i} + \frac{N_j(y_{j+1} - y)}{k_j} + \frac{N_{j+1}(y - y_j)}{k_j} \quad \text{----------------- (1)}$$

Let $$\left.\begin{aligned} x_{i+1} - x_i &= h_i \\ y_{j+1} - y_j &= k_j \end{aligned}\right\} \quad \begin{aligned} &i = 1,2, \text{----}, n-1 \\ &j = 1,2, \text{-----}, m-1 \end{aligned}$$

Integrating (1) with respect to x,

$$\frac{\partial S_{ij}}{\partial x} = -\frac{M_i(x_{i+1} - x)^2}{2h_i} + \frac{M_{i+1}(x - x_i)^2}{2h_i} + \frac{N_j(y_{j+1} - y)(x - x_i)}{k_j} + \frac{N_{j+1}(y - y_j)(x_{i+1} - x)}{k_j} + A(y_{j+1} - y) + B$$

----------------- (2)

Integrating (2) with respect to x,

$$S_{ij} = \frac{M_i(x_{i+1} - x)^3}{6h_i} + \frac{M_{i+1}(x - x_i)^3}{6h_i} + \frac{N_j(y_{j+1} - y)(x - x_i)^2}{2k_j} - \frac{N_{j+1}(y - y_j)(x_{i+1} - x)^2}{2k_j} + \quad \text{---------(3)}$$
$$A(y_{j+1} - y)(x - x_i) + B(x_{i+1} - x) + C(y - y_j) + D$$

Since the spline interpolates at the knots:

(i) $S_{ij}(x_{i+1}, y_j) = z_{i+1,j}$

(ii) $S_{ij}(x_i, y_{j+1}) = z_{i,j+1}$

(iii) $S_{ij}(x_i, y_j) = z_{i,j}$

(iv) $S_{ij}(x_{i+1}, y_{j+1}) = z_{i+1,j+1}$

Applying the conditions (i), (ii), (iii), (iv) to (3)

$$z_{i+1,j} = \frac{M_{i+1}h_i^2}{6} + \frac{N_j h_i^2}{2} + Ah_i k_j + D \qquad \text{------------------- (4)}$$

$$z_{i,j+1} = \frac{M_i h_i^2}{6} - \frac{N_{j+1}h_i^2}{2} + Bh_i + Ck_j + D \qquad \text{------------------- (5)}$$

$$z_{i,j} = \frac{M_i h_i^2}{6} + Bh_i + D \qquad \text{------------------- (6)}$$

$$z_{i+1,j+1} = \frac{M_{i+1}h_i^2}{6} + Ck_j + D \qquad \text{-------------------- (7)}$$

Solving (4), (5), (6), (7) we get the following constants

$$A = \frac{(z_{i+1,j} - z_{ij})}{h_i k_j} - \frac{(z_{i+1,j+1} - z_{i,j+1})}{h_i k_j} + \frac{(N_{j+1} - N_j)h_i}{2k_j}$$

$$B = \frac{(M_{i+1} - M_i)h_i}{6} + \frac{N_{j+1}h_i}{2} - \frac{(z_{i+1,j+1} - z_{i,j+1})}{h_i}$$

$$C = \frac{(z_{i,j+1} - z_{ij})}{k_j} + \frac{N_{j+1}h_i^2}{2k_j}$$

$$D = (z_{i+1,j+1} - z_{i,j+1}) + z_{ij} - \left(\frac{M_{i+1}}{6} + \frac{N_{j+1}}{2}\right)h_i^2$$

So from (3) the two variable cubic spline is

$$\begin{aligned} S_{ij} = {} & \frac{M_i(x_{i+1} - x)^3}{6h_i} + \frac{M_{i+1}(x - x_i)^3}{6h_i} + \frac{N_j(y_{j+1} - y)(x - x_i)^2}{2k_j} - \frac{N_{j+1}(y - y_j)(x_{i+1} - x)^2}{2k_j} \\ & + \left\{\frac{(z_{i+1,j} - z_{ij})}{h_i k_j} - \frac{(z_{i+1,j+1} - z_{i,j+1})}{h_i k_j} + \frac{(N_{j+1} - N_j)h_i}{2k_j}\right\}(y_{j+1} - y)(x - x_i) + \\ & \left\{\frac{(M_{i+1} - M_i)h_i}{6} + \frac{N_{j+1}h_i}{2} - \frac{(z_{i+1,j+1} - z_{i,j+1})}{h_i}\right\}(x_{i+1} - x) + \qquad \text{-------(8)} \\ & \left\{\frac{(z_{i,j+1} - z_{ij})}{k_j} + \frac{N_{j+1}h_i^2}{2k_j}\right\}(y - y_j) + \left\{(z_{i+1,j+1} - z_{i,j+1}) + z_{ij} - \left(\frac{M_{i+1}}{6} + \frac{N_{j+1}}{2}\right)h_i^2\right\}, \end{aligned}$$

$\forall (x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}], \forall i = 1,2,.....,(n-1), \forall j = 1,2,......(m-1)$

In two variable spline function there exist a unique tangent plane at the two surfaces in every node. So corresponding to the node (x_{i+1}, y_j) we have $\frac{\partial S_{ij}}{\partial x}(x_{i+1}, y_j) = \frac{\partial S_{i+1,j}}{\partial x}(x_{i+1}, y_j)$

$$\frac{\partial S_{ij}}{\partial x}(x, y) = -\frac{M_i(x_{i+1} - x)^2}{2h_i} + \frac{M_{i+1}(x - x_i)^2}{2h_i} + \frac{N_j(y_{j+1} - y)(x - x_i)}{k_j} + \frac{N_{j+1}(y - y_j)(x_{i+1} - x)}{k_j}$$

$$+\left\{\frac{(z_{i+1,j}-z_{ij})}{h_i k_j}-\frac{(z_{i+1,j+1}-z_{i,j+1})}{h_i k_j}+\frac{(N_{j+1}-N_j)h_i}{2k_j}\right\}(y_{j+1}-y)$$

$$-\left\{\frac{(M_{i+1}-M_i)h_i}{6}+\frac{N_{j+1}h_i}{2}-\frac{(z_{i+1,j+1}-z_{i,j+1})}{h_i}\right\}$$

$$\frac{\partial S_{ij}}{\partial x}(x_{i+1},y_j)=\frac{M_{i+1}h_i}{3}+\frac{N_j h_i}{2}+\frac{z_{i+1,j}-z_{ij}}{h_i}+\frac{M_i h_i}{6} \quad \text{------------------- (9)}$$

$$\frac{\partial S_{i+1,j}}{\partial x}(x,y)=-\frac{M_{i+1}(x_{i+2}-x)^2}{2h_{i+1}}+\frac{M_{i+2}(x-x_{i+1})^2}{2h_{i+1}}+\frac{N_j(y_{j+1}-y)(x-x_{i+1})}{k_j}+$$

$$\frac{N_{j+1}(y-y_j)(x_{i+1}-x)}{k_j}+\left\{\frac{(z_{i+1,j}-z_{ij})}{h_i k_j}-\frac{(z_{i+1,j+1}-z_{i,j+1})}{h_i k_j}+\frac{(N_{j+1}-N_j)h_i}{2k_j}\right\}(y_{j+1}-y)$$

$$-\left\{\frac{(M_{i+1}-M_i)h_i}{6}+\frac{N_{j+1}h_i}{2}-\frac{(z_{i+1,j+1}-z_{i,j+1})}{h_i}\right\}$$

$$\frac{\partial S_{i+1,j}}{\partial x}(x_{i+1},y_j)=-\frac{M_{i+1}}{3}h_{i+1}+\frac{z_{i+2,j}}{h_{i+1}}-\frac{z_{i+1,j}}{h_{i+1}}-\frac{M_{i+2}}{6}h_{i+1}-\frac{N_j}{2}h_{i+1} \quad \text{----------------------- (10)}$$

Equating (9) and (10)

$$\frac{(z_{i+2,j}-z_{i+1,j})}{h_{i+1}}-\frac{(z_{i+1,j}-z_{ij})}{h_i}=\frac{M_{i+1}}{3}(h_i+h_{i+1})+\frac{N_j}{2}(h_i+h_{i+1})+\frac{M_i h_i}{6}+\frac{M_{i+2}}{6}h_{i+1} \quad \text{--------(11)}$$

Assume $\frac{z_{i+1,j}-z_{ij}}{h_i}=\sigma_{i,j}$ where i = 1,2, ----,(n-1) and j = 1,2,-----,(m-1)

$$\frac{z_{i+2,j}-z_{i+1,j}}{h_{i+1}}=\sigma_{i+1,j} \quad \text{----------------- (12)}$$

Assume $\lambda_i=\frac{h_{i+1}}{h_i+h_{i+1}}$ where i = 1,2, ----,(n-2) and j = 1,2,-----,(m-2)

$$\mu_i=1-\lambda_i=\frac{h_i}{h_i+h_{i+1}} \quad \text{----------------- (13)}$$

$$d_{ij}=\frac{6(\sigma_{i+1,j}-\sigma_{i,j})}{h_i+h_{i+1}}$$

Therefore equation (11) will become

$$6[\sigma_{i+1,j}-\sigma_{i,j}]=2M_{i+1}(h_i+h_{i+1})+3N_j(h_i+h_{i+1})+M_i h_i+M_{i+2}h_{i+1} \quad \text{---------------(14)}$$

Dividing equation (14) by h_i+h_{i+1} we get

$2M_{i+1}+3N_j+\mu_i M_i+\lambda_i M_{i+2}=d_{ij}$,wherei = 1,2, ----,(n-2) and j = 1,2,-----,(m-2) -----------(15)

So the two variable natural cubic spline is

$$S_{ij} = \frac{M_i(x_{i+1}-x)^3}{6h_i} + \frac{M_{i+1}(x-x_i)^3}{6h_i} + \frac{N_j(y_{j+1}-y)(x-x_i)^2}{2k_j} - \frac{N_{j+1}(y-y_j)(x_{i+1}-x)^2}{2k_j}$$
$$+\left\{\frac{(z_{i+1,j}-z_{ij})}{h_ik_j} - \frac{(z_{i+1,j+1}-z_{i,j+1})}{h_ik_j} + \frac{(N_{j+1}-N_j)h_i}{2k_j}\right\}(y_{j+1}-y)(x-x_i)+$$
$$\left\{\frac{(M_{i+1}-M_i)h_i}{6} + \frac{N_{j+1}h_i}{2} - \frac{(z_{i+1,j+1}-z_{i,j+1})}{h_i}\right\}(x_{i+1}-x)+$$
$$\left\{\frac{(z_{i,j+1}-z_{ij})}{k_j} + \frac{N_{j+1}h_i^2}{2k_j}\right\}(y-y_j)+\left\{(z_{i+1,j+1}-z_{i,j+1})+z_{ij}-\left(\frac{M_{i+1}}{6}+\frac{N_{j+1}}{2}\right)h_i^2\right\},$$

$$\forall(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}], \forall i=1,2,.....,(n-1), \forall j=1,2,......(m-1)$$

where $2M_{i+1}+3N_j+\mu_iM_i+\lambda_iM_{i+2}=d_{ij}$, i = 1,2, ----,(n-2) and j = 1,2,-----,(m-2)

For natural spline $M_1 = M_n = N_1 = N_m = 0$

III. Illustration

Consider the two variable function $f(x,y)=e^{xy}$. The following table gives the function values for x taking values 0, 0.1, 0.2 and y taking values 0, 0.1, 0.2

Y / X	0	0.1	0.2
0	1	1	1
0.1	1	1.01	1.02
0.2	1	1.02	1.04

$h_1 = h_2 = 0.1$

$k_1 = k_2 = 0.1$

Using (12) and (13),

$\sigma_{11} = 0$

$\sigma_{12} = 0.1$ $\lambda_1 = 0.5$ $\mu_1 = 0.5$ $d_{11} = 0$

$\sigma_{21} = 0$ $d_{12} = 0.5$

$\sigma_{22} = 0.1$

Using (15) $$\begin{aligned}2M_2+3N_1+\mu_1M_1+\lambda_1M_3=d_{11}\\2M_2+3N_2+\mu_1M_1+\lambda_1M_3=d_{12}\end{aligned} \quad \text{------------- (16)}$$

For a natural cubic spline $M_1 = M_3 = N_1 = N_3 = 0$

Solving (16) $M_2 = 0$ and $N_2 = 0.1667$

Using (8) the two variable cubic splines are

$$S_{ij}(x,y)=\begin{cases} -0.8335(y)(0.1-x)^2-0.9167(0.1-y)(x)-0.09167(0.1-x)+0.00834(y)+1.0092 \\ \qquad (x,y)\in I_{11} \\ 0.8335(0.2-y)(x)^2-1.0834(0.2-y)(x)-0.2(0.1-x)+1.04, (x,y)\in I_{12} \\ -0.8335(y)(0.2-x)^2-0.9167(0.1-y)(x-0.1)-0.09167(0.2-x)+0.10834y+1.0092 \\ \qquad (x,y)\in I_{21} \\ 0.8335(0.2-y)(x-0.1)^2-1.0834(0.2-y)(x-0.1)-0.2(0.2-x)+0.1(y-0.1)+1.03 \\ \qquad (x,y)\in I_{22}\end{cases}$$

The following table shows the values of the function f at (x, y) and the corresponding interpolated values $S_{ij}(x,y)$

	(x, y)	$S_{ij}(x,y)$	$f(x,y)$	**Error**
1	(0.05,0)	1	1	0
2	(0.05,0.05)	1.003	1.003	0
3	(0.15,0)	1	1	0
4	(0.15,0.15)	1.022	1.022	0
5	(0.05,0.15)	1.007	1.007	0

IV. Conclusion

Interpolated values are found to be same as the actual functional value.

References

[1]. Anthony Ralston and Philip Rabinowitz, A first course in Numerical Analysis, Second Edition, Dover publication , INC, New York.

[2]. Ion Lixandru, Algorithm for the calculation of the two variables cubic spline function,AnaleleStiintifice Ale UniversitatiiMatematica, Tomul Lix,2013,f.1

[3]. Richard L Burden, J. Douglas Faires, Numerical Analysis, Ninth Edition,Brooks/Cole Cengage Learning.

[4]. S.S. Sastry , Introductory Methods Of Numerical Analysis, Fourth Edition, PHI Learning Private Limited