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$\mathbf{I_2}$ − STATISTICAL And $\mathbf{I_2}$ −Lacunary Statistical Convergence for Double Sequence of Order α

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Abstract: *Following the recent generalization of the recently introduced summability methods, namely I −statistical convergence of order α and I −lacunary statistical convergence of order α, where $0 < \alpha < 1$; Relationships were investigated and some observations about these classes were made and answers were proffered to the open problems posed by Das, Savas and Ghosal in [10]. We shall analogously extend above notions to double sequences.*
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I. Introduction

The concept of statistical convergence was formally introduced by Fast [12] and Schoenberg [24] independently. Although statistical convergence was introduced over fifty years ago, it has become an active area of research in recent years. This has been applied in various areas such as summability theory (Fridy [13] and Salat [22]), topological groups (Cakalli [2], [3]), topological spaces (Maio and Kocinac [16]), locally convex spaces (Maddox [17]), measure theory (Cheng et al [4]), (Connor and Swardson [7]) and (Miller [19]), Fuzzy Mathematics (Nuray and Savas [21] and Savas [23]). In recent years generalization of statistical convergence has appeared in the study of strong summability and the structure of ideals of bounded functions, (Connor and Swardson [7]). Salat et al [15]) further extended the idea of statistical convergence to I −convergence using the notion of ideals of $\mathbb{N}$ with many interesting consequences. Fridy and Orhan [14] introduced in another direction a new type of convergence called lacunary statistical convergence. Das and Savas [11] introduced and studied I −statistical and I −lacunary statistical convergence of order α. In this paper in analogy to Das and Savas [11], we shall introduce and study I_2 −statistical and I_2 −lacunary statistical convergence of order α as follows:

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n,m)$ be the numbers and (i,j) in K such that $i \leq n$ and $j \leq m$. Then the two dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as:

$$\underline{\delta_2}(K) = \lim_{n,m} \inf \frac{K(n,m)}{nm}$$

In case the sequence $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense, then we say that K has a double natural density and is defined by

$$\lim_{n,m} \inf \frac{K(n,m)}{nm} = \delta_2(K)$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$.

$$\delta_2(K) = \lim_{n,m} \inf \frac{K(n,m)}{nm} = \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

i.e, the set K has double natural density zero, while the set $K = \{(i, 2j) : i, j \in \mathbb{N}\}$ has double natural density ½.

Note that if $n = m$, we have a two-dimensional natural density considered by Christopher [5].

Statistical convergence of double sequences $x = (x_{jk})$ is defined as follows:

Definition 1.1 (Mursaleen and Edely [20]): A real double sequence $x = (x_{jk})$ is statistically convergent to a number L if for each $\varepsilon > 0$, the set

$$\{(j,k), j \leq n, k \leq m : |x_{jk} - L| \geq \varepsilon\}$$

has double natural density zero. In this case, we write $st_2 - \lim_{jk} x_{jk} = L$ and we denote the set of all statistically convergent double sequences by st_2.

II. I −Statistical And I −Lacunary Statistical Convergence of Order α

The following definitions and results are by Das and Savas [11]

Definition 2.1: If X is a non-empty set then a family of set $I \subset P(X)$ is called an ideal in X if and only if (i) $\Phi \in I$; (ii) For each $A, B \in I$ we have $A \cup B \in I$; (iii) For each $A \in I$ and $B \subset A$ we have $B \in I$.

Definition 2.2: Let X is a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a filter on X if and only if (i) $\Phi \notin F$; (ii) For each $A, B \in F$ we have $A \cap B \in F$; (iii) For each $A \in F$ and $B \supset A$ we have $B \in F$.

An ideal I is called non-trivial if $I \neq \Phi$ and $X \notin I$.

Definition 2.3: A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons, i.e., if it contains $\{\{x\}: x \in X\}$.

For further study we shall take $X = \mathbb{N}^2$ and I will denote an ideal of subsets of $\mathbb{N}^2$. The following proposition express a relation between the notions of an ideal and a filter.

Proposition 2.1: Let $I \subset P(\mathbb{N}^2)$ be a non- trivial ideal. Then the class $F = F(I) = \{M \subset \mathbb{N}^2: M = \mathbb{N}^2 - A, for\ some\ A \in I\}$ is a filter on $\mathbb{N}^2$ (we shall call $F = F(I)\ the\ filter\ associated\ with\ I$).

Definition 2.4: Let $I \subset P(\mathbb{N}^2)$ be a non-trivial ideal in $\mathbb{N}^2$. A double sequence $x = (x_{jk})$ of real numbers is said to be I −convergent to a number L if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{(i,j) \in \mathbb{N}^2: |x_{jk} - L| \geq \varepsilon\}$ belongs to I.The number L is called the I −limit of the sequence (x_{ij}) and we write $I - \lim_{jk} x_{jk} = L$.

Remark 2.1: If we take $I = \{E \subset \mathbb{N}^2: E\ is\ contained\ (\mathbb{N} \times A) \cup (A \times \mathbb{N}) where\ A\ is\ a\ finite\ subset\ of\ N\}$. Then $I -$ convergent is equivalent to the usual Pringsheim's convergence.

Definition 2.5: A double sequence $x = (x_{jk})$ is said to be convergent to L in the Pringsheim's sense (1900) if for each $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k \geq m$. The number L is called the Pringsheim limit of the sequence x.

Definition 2.6: A double sequence $x = (x_{jk})$ is said to be bounded if there exists a real number $M > 0$ such that $|x_{jk}| < M$ for each i and$j$, i.e., if $\|x\|_{(\infty,2)} = \sup_{jk} |x_{jk}| < \infty$. We shall denote the set of all bounded double sequences by ℓ_∞^2. Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.

Definition 2.7 (Das and Savas [11]): A sequence $x = (x_k)_{k \in \mathbb{N}}$ is said to be I −statistically convergent of order α to L or $S(I)^\alpha$ −convergent to L, where $0 < \alpha \leq 1$, if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{k \in \mathbb{N}: \frac{1}{n^\alpha} |\{k \leq n: |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in I.$$

In this case we write $x_k \to L(S(I)^\alpha)$. The class of all sequences that are I −statistically convergent of order α will be denoted by $S(I)^\alpha$.

Remark 2.2: For $I = I_f, S(I)^\alpha$ −convergence coincides with statistical convergence of order α(see Colak [6]). For an arbitrary ideal I and for $\alpha = 1$ it coincides with I −statistical convergence (see Das et al [10]). When $I = I_f$ and $\alpha = 1$ it reduces to statistical convergence.

Definition 2.8 (Das and Savas [11]): Let θ be a lacunary sequence. A sequence $x = (x_k)_{k \in \mathbb{N}}$ is said to be I −lacunary statistically convergent of order α to L or $S_\theta(I)^\alpha$ −convergent to L if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{r \in \mathbb{N}: \frac{1}{h_r^\alpha} |\{k \in I_r: |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in I.$$

In this case we write $x_k \to L(S_\theta(I)^\alpha)$. The class of all I −lacunary statistically convergent sequences of order α will be denoted by $S_\theta(I)^\alpha$.

Remark 2.3: For $\alpha = 1$ the definition coincides with I −lacunary statistical convergence (see Das et al [10]).

Theorem 2.1 (Das and Savas [11]): Let $0 < \alpha \leq \beta \leq 1$. Then $S(I)^\alpha \subset S(I)^\beta$ and the inclusion is strict for at least those α, β for which there is a $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$ and when $I = I_{fin}$.

Theorem 2.2 (Das and Savas [11]): $S(I)^\alpha \cap l_\infty$ is a closed subset of l_∞; where $l_\infty = \{x \in \omega: \sup_k |x_k| < \infty\}$.

Definition 2.9 (Das and Savas [11]): Let θ be a lacunary sequence. Then $x = (x_k)_{k \in \mathbb{N}}$ is said to be $N_\theta(I)^\alpha$ −convergent to L if for any $\varepsilon > 0$

$\left\{r \in \mathbb{N}: \frac{1}{h_r^\alpha} \sum_{k \in I_r} |x_k - L| \geq \varepsilon\right\} \in I.$

It is denoted by $x_k \to L(N_\theta(I)^\alpha)$ and the class of such sequences will be denoted simply by $N_\vartheta(I)^\alpha$.

Theorem 2.3 (Das and Savas [11]): Let $\theta = \{k_r\}_{r \in \mathbb{N}}$ be a lacunary sequence. Then

(a) $x_k \to L(N_\theta(I)^\alpha) \Longrightarrow x_k \to L(S_\theta(I)^\alpha)$, and

(b) $N_\theta(I)^\alpha$ is a proper subset of $S_\theta(I)^\alpha$

Theorem 2.4 (Das and Savas [11]): For any lacunary sequence θ, I −statistical convergence of order α implies I −lacunary statistical convergence of order α if $\lim_r inf q_r^\alpha > 1$.

Theorem 2.5 (Das and Savas [11]): For a lacunary sequence θ satisfying the above condition, I −lacunary statistical convergence implies I −statistical convergence if $\lim_r supq_r < \infty$.
Theorem 2.6 (Das and Savas [11]): For a lacunary sequence θ satisfying the above condition, I −lacunary statistical convergence of order α implies I −statistical convergence of order $\alpha, 0 < \alpha < 1$, if

$$\sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^{\alpha}}{(k_{r-1})^{\alpha}} = B < \infty.$$

III. I_2 −Statistical And I_2 −Lacunary Statistical Convergence For Double Sequence Of Order α

We now introduce our main definitions and results.
Definition 3.1: A double sequence $x = (x_{jk})_{j,k\in\mathbb{N}}$ is said to be I_2 −statistically convergent of order α to L or$S(I_2)^{\alpha}$ −convergent to L, where $0 < \alpha \le 1$, if for each $\varepsilon > 0$ and $\delta > 0$
$\left\{(n,m) \in \mathbb{N}\times\mathbb{N}: \frac{1}{(nm)^{\alpha}} \left|\{j \le m, k \le n: |x_{jk} - L| \ge \varepsilon\}\right| \ge \delta\right\} \in I_2$.
In this case we write $x_{jk} \to L(S(I_2)^{\alpha})$. The class of all sequences that are I_2 −statistically convergent of order α will be denoted simply by $S(I_2)^{\alpha}$.
Remark 3.1: For an arbitrary ideal I_2 and for $\alpha = 1$ it coincides with I_2 −statistical convergence of double sequence. When $I_2 = I_{2f}$ and $\alpha = 1$ it reduces to statistical convergence of double sequence.
Definition 3.2: Let $\theta_{r,s}$ be a lacunary sequence. A double sequence $x = (x_{jk})_{j,k\in\mathbb{N}}$ is said to be I_2 −lacunary statistical convergent of order α to L or $S_{\theta_{r,s}}(I_2)^{\alpha}$ −convergent to L if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}} \left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \varepsilon\}\right| \ge \delta\right\} \in I_2$$

In this case we write $x_{jk} \to L(S_{\theta_{r,s}}(I_2)^{\alpha})$. The class of all I_2 −lacunary statistically convergent sequences of order α will be denoted by $S_{\theta_{r,s}}(I_2)^{\alpha}$.
Remark 3.2: For $\alpha = 1$ the definition coincides with I_2 −lacunary statistical convergent of double sequence.
Theorem 3.1: Let $0 < \alpha \le \beta \le 1$. Then $S(I_2)^{\alpha} \subset S(I_2)^{\beta}$ and the inclusion is strict for at least those α, β for which there are $j, k \in \mathbb{N}$ such that $\alpha < \frac{1}{jk} < \beta$ and when $I_2 = I_{2f}$.
Proof: $S_{\theta_{r,s}}(I_2)$ for all $n = 1,2,3,\ldots.x^n$ is I_2 −statistically convergent to some number L_n for $n = 1,2,3,\ldots$. We shall first show that the sequence $\{L_n\}_{n\in\mathbb{N}}$ is convergent to some number L and the sequence $x = (x_{jk})_{j,k\in\mathbb{N}\times\mathbb{N}}$ is I_2 −statistically convergent of order α to L. Take a strictly decreasing sequence of positive numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$ converging to 0. Choose a positive integer n such that
$\|x - x^n\|_{\infty} < \frac{\epsilon_n}{4}$. Let $0<\delta < 1$. Then

$$A = \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}} \left|\left\{j,k \in I_{r,s}: |x_{jk}^n - L_n| \ge \frac{\epsilon_n}{4}\right\}\right| < \frac{\delta}{3}\right\} \in F(I_2)$$

and $B = \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}} \left|\left\{j,k \in I_{r,s}: |x_{jk}^{n+1} - L_{n+1}| \ge \frac{\epsilon_{n+1}}{4}\right\}\right| < \frac{\delta}{3}\right\} \in F(I_2)$.
Since $A \cap B \in F(I_2)$ and $\phi \notin F(I_2)$, so we can choose $r \in A \cap B$. Then

$$\frac{1}{h_{r,s}} \left|\left\{j,k \in I_{r,s}: |x_{jk}^n - L_n| \ge \frac{\epsilon_n}{4}\right\}\right| < \frac{\delta}{3}$$

and

$$\frac{1}{h_{r,s}} \left|\left\{j,k \in I_{r,s}: |x_{jk}^{n+1} - L_{n+1}| \ge \frac{\epsilon_{n+1}}{4}\right\}\right| < \frac{\delta}{3}$$

and so $\frac{1}{h_{r,s}} \left|\left\{j,k \in I_{r,s}: |x_{jk}^n - L_n| \ge \frac{\epsilon_n}{4} \vee |x_{jk}^{n+1} - L_n| \ge \frac{\epsilon_{n+1}}{4}\right\}\right| < \delta < 1$.
Hence there exist $j,k \in I_{r,s}$ for which $|x_{jk}^n - L_n| < \frac{\epsilon_n}{4}$ and $|x_{jk}^{n+1} - L_{n+1}| < \frac{\epsilon_{n+1}}{4}$. Then we can write

$$\begin{aligned}|L_n - L_{n+1}| &< |L_n - x_{jk}^n| + |x_{jk}^n - x_{jk}^{n+1}| + |x_{jk}^{n+1} - L_{n+1}| \\ &\le |x_{jk}^n - L_n| + |x_{jk}^{n+1} - L_{n+1}| + \|x - x^n\|_{\infty} + \|x - x^{n+1}\|_{\infty} \le \frac{\epsilon_n}{4} + \frac{\epsilon_{n+1}}{4} + \frac{\epsilon_n}{4} + \frac{\epsilon_{n+1}}{4} < \epsilon_n.\end{aligned}$$

This implies that $\{L_n\}_{n\in\mathbb{N}}$
Is a Cauchy sequence in $\mathbb{R}$ and so there is a real number L such that $L_n \to L$ as $n \to \infty$. We need to prove that$x \to L(S_{\theta_{r,s}}(I_2)^{\alpha})$. For any $\epsilon > 0$, choose $n \in \mathbb{N}$ suh that $\epsilon_n < \frac{\epsilon}{4}$, $\|x - x^n\|_{\infty} < \frac{\epsilon}{4}$, $|L_n - L| < \frac{\epsilon}{4}$. Then

$$\begin{aligned}\frac{1}{h_{r,s}^{\alpha}} \left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| &\le \frac{1}{h_{r,s}^{\alpha}} \left|\left\{j,k \in I_{r,s}: |x_{jk}^n - L_n| + \|x_{jk} - x_{jk}^n\|_{\infty} + |L_n - L| \ge \epsilon\right\}\right| \\ &\le \frac{1}{h_{r,s}^{\alpha}} \left|\left\{j,k \in I_{r,s}: |x_{jk}^n - L_n| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \ge \epsilon\right\}\right| \le \frac{1}{h_{r,s}^{\alpha}} \left|\left\{j,k \in I_{r,s}: |x_{jk}^n - L_n| \ge \frac{\epsilon}{2}\right\}\right|.\end{aligned}$$

This implies $\left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\left\{j,k \in I_{r,s}: \left|x_{jk} - L\right| \geq \epsilon\right\}\right| < \delta\right\} \supseteq \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\left\{j,k \in I_{r,s}: \left|x_{jk}^{n} - L_n\right| \geq \right.\right.\right.$ *ϵ2<δ∈FI2* and so *r,s∈N×N:1hr,sαj,k∈Ir,s:xjk−L≥ϵ<δ∈I2.* This establishes the fact that *x→LSθr,s(I2)α* which completes the proof of the theorem.

Theorem 3.2: $S(I_2)^{\alpha} \cap l_{\infty}^{2}$ is a closed subset of l_{∞}^{2}; where $l_{\infty}^{2} = \left\{x \in \omega: \sup_{j,k}\left|x_{jk}\right| < \infty\right\}$.

Definition 3.3: Let $\theta_{r,s}$ be a double lacunary sequence. Then $x = (x_{jk})_{j,k\in\mathbb{N}\times\mathbb{N}}$ is said to be $N_{\theta_{r,s}}(I_2)^{\alpha}$ −convergent to L if for any $\epsilon > 0$

$$\left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}} \sum_{j,k\in I_{r,s}} \left|x_{jk} - L\right| \geq \epsilon\right\} \in I_2.$$

It is denoted by $x_{jk} \to L\left(N_{\theta_{r,s}}(I_2)^{\alpha}\right)$ and the class of such double sequences will be denoted simply by $N_{\theta_{r,s}}(I_2)^{\alpha}$.

Theorem 3.3: Let $\theta_{r,s} = \{k_{r,s}\}_{r,s\in\mathbb{N}}$ be a double lacunary sequence. Then

(a) $x_{jk} \to L\left(N_{\theta_{r,s}}(I_2)^{\alpha}\right) \Rightarrow x_{jk} \to L\left(S_{\theta_{r,s}}(I_2)^{\alpha}\right)$, and

(b) $N_{\theta_{r,s}}(I_2)^{\alpha}$ is a proper subset of $S_{\theta_{r,s}}(I_2)^{\alpha}$.

Proof: (a) If $\epsilon > 0$ and $x_{jk} \to L\left(N_{\theta_{r,s}}(I_2)^{\alpha}\right)$, we can write $\sum_{j,k\in I_{r,s}}\left|x_{jk} - L\right| \geq \sum_{j,k\in I_{r,s} |x_{jk}-L|\geq\epsilon}\left|x_{jk} - L\right| \geq \epsilon\left|\left\{j,k \in I_{r,s}: \left|x_{jk} - L\right| \geq \epsilon\right\}\right|$ and so

$$\frac{1}{\epsilon \cdot h_{r,s}^{\alpha}} \sum_{j,k\in I_{r,s}} \left|x_{jk} - L\right| \geq \frac{1}{h_{r,s}^{\alpha}}\left|\left\{j,k \in I_{r,s}: \left|x_{jk} - L\right| \geq \epsilon\right\}\right|.$$

Then for any $\delta > 0$

$$\left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\left\{j,k \in I_{r,s}: \left|x_{jk} - L\right| \geq \epsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}} \sum_{j,k\in I_{r,s}} \left|x_{jk} - L\right| \geq \epsilon\cdot\delta\right\} \in I_2.$$

This proves the result.

(b) In order to establish that the inclusion $N_{\theta_{r,s}}(I_2)^{\alpha} \subset S_{\theta_{r,s}}(I_2)^{\alpha}$ is proper, let $\theta_{r,s}$ be given and define x_{jk} to be 1,2,...,$\left[\sqrt{h_{r,s}^{\alpha}}\right]$ at the first $\left[\sqrt{h_{r,s}^{\alpha}}\right]$ integers in $I_{r,s}$ and $x_{jk} = 0$ otherwise for all $r = s = 1,2,3,\ldots$. Then for any $\epsilon > 0$,

$$\frac{1}{h_{r,s}^{\alpha}}\left|\left\{j,k \in I_{r,s}: \left|x_{jk} - 0\right| \geq \epsilon\right\}\right| \leq \frac{\left[\sqrt{h_{r,s}^{\alpha}}\right]}{h_{r,s}^{\alpha}}$$

and for any $\delta > 0$ we get

$$\left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\left\{j,k \in I_{r,s}: \left|x_{jk} - 0\right| \geq \epsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{\left[\sqrt{h_{r,s}^{\alpha}}\right]}{h_{r,s}^{\alpha}} \geq \delta\right\}.$$

Since the set on the right hand side is finite and so belongs to I, it follows that $x_{jk} \to 0\left(S_{\theta_{r,s}}(I_2)^{\alpha}\right)$. On the other hand

$$\frac{1}{h_{r,s}^{\alpha}}\sum_{j,k\in I_{r,s}}\left|x_{jk} - 0\right| = \frac{1}{h_{r,s}^{\alpha}} \cdot \frac{\left[\sqrt{h_{r,s}^{\alpha}}\right]\left(\left[\sqrt{h_{r,s}^{\alpha}}\right]\right)}{2}$$

Then

$$\left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}} \sum_{j,k\in I_{r,s}} \left|x_{jk} - 0\right| \geq \frac{1}{4}\right\} = \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{\left[\sqrt{h_{r,s}^{\alpha}}\right]\left(\left[\sqrt{h_{r,s}^{\alpha}}\right] + 1\right)}{h_{r,s}} \geq \frac{1}{2}\right\}$$
$$= \{m, m+1, m+2, m+3, \ldots\}$$

For some $m \in \mathbb{N}$ which belongs to $F(I_2)$ since I_2 is admissible for double sequences. So

$$x_{jk} \nrightarrow 0\left(N_{\theta_{r,s}}(I_2)^{\alpha}\right).$$

Theorem 3.4: For any lacunary double sequence $\theta_{r,s}$, I_2 −statistical convergence of order α implies I_2 −lacunary statistical convergence of order α if $lim\,\inf_{r,s} q_{r,s}^{\alpha} > 1$.

Proof: Suppose first that $lim\,\inf_{r,s} q_{r,s}^{\alpha} > 1$. Then there exists $\sigma > 0$ such that $q_{r,s}^{\alpha} \geq 1 + \sigma$ for sufficiently large r and s which implies that

$$\frac{h_{r,s}^{\alpha}}{l_{r,s}^{\alpha}} \geq \frac{\sigma}{1+\sigma}.$$

Since $x_{jk} \to L\left(S_{\theta_{r,s}}(I_2)^{\alpha}\right)$, then for every $\epsilon > 0$ and for sufficiently large r and s, we have

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$$\frac{1}{l_{r,s}^{\alpha}}\left|\{l \le l_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \ge \frac{1}{l_{r,s}^{\alpha}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \ge \frac{\sigma}{1+\sigma}\frac{1}{h_{r,s}^{\alpha}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right|.$$

Then for any $\delta > 0$, we get

$$\left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \ge \delta\right\} \subseteq \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{l_{r,s}^{\alpha}}\left|\{l \le l_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \ge \frac{\delta\sigma}{(1+\sigma)}\right\} \in I_2.$$

This proves the result.

Theorem 2.5: For lacunary double sequence $\theta_{r,s}$ satisfying the above condition, I_2 −lacunary statistical convergence implies I_2 −statistical convergence if $lim\sup_{r,s} q_{r,s} < \infty$.

Proof: If $lim\sup_{r,s} q_{r,s} < \infty$. Then without any loss of generality we can assume that there exists a $0 < B < \infty$ such that $q_{r,s} < B$ for all $r,s \ge 1$. Suppose that $x_{jk} \to L(S_{\theta_{r,s}}(I_2))$ and for $\epsilon, \delta, \delta_1 > 0$ define the set

$$C = \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \ge \delta\right\}$$

and

$$T = \left\{(n,m) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\left|\{j \le m, k \le n: |x_{jk} - L| \ge \epsilon\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in F(I_2)$, the filter associated with the ideal I_2. Further observe that

$$A_{i,j} = \frac{1}{h_{i,j}}\left|\{l,k \in I_{i,j}: |x_{lk} - L| \ge \epsilon\}\right| < \delta$$

For all $i,j \in C$. Let $(n,m) \in \mathbb{N}\times\mathbb{N}$ be such that $k_{r-1} < n < k_r$ and $l_{s-1} < m < l_s$ for some $r,s \in C$. Now

$$\begin{aligned}
&\frac{1}{mn}\left|\{k \le n, l \le m: |x_{jk} - L| \ge \epsilon\}\right| < \delta \le \frac{1}{k_{r-1}l_{s-1}}\left|\{k \le k_r, l \le l_s: |x_{jk} - L| \ge \epsilon\}\right| \\
&= \frac{1}{k_{r-1}l_{s-1}}\left|\{j,k \in I_{1,1}: |x_{jk} - L| \ge \epsilon\}\right| + \cdots + \frac{1}{k_{r-1}l_{s-1}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \\
&= \frac{k_1}{k_{r-1}}\frac{l_1}{l_{s-1}}\frac{1}{h_{1,1}}\left|\{j,k \in I_{1,1}: |x_{jk} - L| \ge \epsilon\}\right| + \frac{k_2-k_1}{k_{r-1}}\frac{l_2-l_1}{l_{s-1}}\frac{1}{h_{2,2}}\left|\{j,k \in I_{2,2}: |x_{jk} - L| \ge \epsilon\}\right| \\
&+ \frac{k_r-k_{r-1}}{k_{r-1}}\frac{l_s-l_{s-1}}{l_{s-1}}\frac{1}{h_{r,s}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \\
&= \frac{k_1}{k_{r-1}}\frac{l_1}{l_{s-1}}A_{11} + \frac{k_2-k_1}{k_{r-1}}\frac{l_2-l_1}{l_{s-1}}A_{22} + \cdots + + \frac{k_r-k_{r-1}}{k_{r-1}}\frac{l_s-l_{s-1}}{l_{s-1}}A_{rs} \le \sup_{i,j\in C} A_{ij}\frac{k_r}{k_{r-1}}\frac{l_s}{l_{s-1}} \\
&< B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\cup\{n,m: k_{r-1} < n < k_r \text{ and } l_{s-1} < m < l_s,\ r,s \in C\} \subset T$ where $C \in F(I_2)$, it follows from our assumption on $\theta_{r,s}$ that the set T also belongs to $F(I_2)$, and this completes the proof of the theorem.

Theorem 3.6: For a lacunary double sequence $\theta_{r,s}$ satisfying the above condition, I_2 −lacunary statistical convergence of order α implies I_2 −statistical convergence of order α, 0< $\alpha < 1$, if

$$\operatorname*{Sup}_{r,s}\sum_{i,j=0,0}^{r-1,s-1}\frac{h_{i+1,j+1}^{\alpha}}{(k_{r-1}l_{s-1})^{\alpha}} = B < \infty$$

Proof: Suppose that $x_{jk} \to L(S_{\theta_{r,s}}(I_2)^{\alpha})$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{r,s \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| < \delta\right\}$$

and

$$T = \left\{(n,m) \in \mathbb{N}\times\mathbb{N}: \frac{1}{(nm)^{\alpha}}\left|\{j \le m, k \le n: |x_{jk} - L| \ge \epsilon\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in F(I_2)$, the filter associated with the ideal I_2. Further observe that

$$A_{ij} = \frac{1}{h_{ij}^{\alpha}}\left|\{l,k \in I_{ij}: |x_{jk} - L| \ge \epsilon\}\right| < \delta$$

For all $i,j \in C$. Let $(n,m) \in \mathbb{N}\times\mathbb{N}$ be such that $k_{r-1} < n < k_r$ and $l_{s-1} < m < l_s$ for some $r,s \in C$.
Now

$$\begin{aligned}
\frac{1}{(mn)^{\alpha}}\left|\{k \le n, j \le m: |x_{jk} - L| \ge \epsilon\}\right| &\le \frac{1}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}}\left|\{k \le k_r, l \le l_s: |x_{jk} - L| \ge \epsilon\}\right| \\
&= \frac{1}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}}\left|\{j,k \in I_{1,1}: |x_{jk} - L| \ge \epsilon\}\right| + \cdots + \frac{1}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \\
&= \frac{k_1^{\alpha} l_1^{\alpha}}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}}\frac{1}{h_{1,1}^{\alpha}}\left|\{j,k \in I_{1,1}: |x_{jk} - L| \ge \epsilon\}\right| \\
&+ \frac{(k_2 - k_1)^{\alpha}(l_2 - l_1)^{\alpha}}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}}\frac{1}{h_{2,2}^{\alpha}}\left|\{j,k \in I_{2,2}: |x_{jk} - L| \ge \epsilon\}\right| + \cdots \\
&+ \frac{(k_r - k_{r-1})^{\alpha}(l_s - l_{s-1})^{\alpha}}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}}\frac{1}{h_{r,s}^{\alpha}}\left|\{j,k \in I_{r,s}: |x_{jk} - L| \ge \epsilon\}\right| \\
&= \frac{k_1^{\alpha} l_1^{\alpha}}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}} A_{1,1} + \frac{(k_2 - k_1)^{\alpha}(l_2 - l_1)^{\alpha}}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}} A_{2,2} + \cdots + \frac{(k_r - k_{r-1})^{\alpha}(l_s - l_{s-1})^{\alpha}}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}} A_{r,s} \\
&\le \operatorname*{Sup}_{i,j \in C} A_{i,j} \operatorname*{Sup}_{r,s} \sum_{i,j=0,0}^{r-1,s-1} \frac{(k_{i+1} - k_i)^{\alpha}(l_{j+1} - l_j)^{\alpha}}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}} < B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\cup \{n, m: k_{r-1} < n < k_r, l_{s-1} < m < l_s, r, s \in C\} \subset T$ where $C \in F(I_2)$, it follows from our assumption on $\theta_{r,s}$ that the set T also belongs to $F(I_2)$ and this completes the proof of the theorem.

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