IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 2 Ver. I (Mar. - Apr. 2017), PP 27-30
www.iosrjournals.org

Pathway Fractional Integral Operator of the Product of Two Aleph - Functions

[1]Dr. Sanjay Bhatter, [2]Mr. Rakesh Kumar Bohra
[1]*Department of Mathematics Malaviya National Institute of Technology Jaipur, 302017, India*
[2]*Department of Mathematics Global College of Technology, Jaipur, 302022, India*

Abstract: *To study a pathway fractional integral operator associated with the pathway model and pathway probability density is the object of present paper. We establish new results on applying the saigo-Maeda operators to the product of two-variable Aleph-function.*
Mathematics Subject Classification: *MSC2010, 11S23, 33C45, 33B15.*
Keywords: *Pathway Fractional integrals operator, Aleph function* ($\aleph$-function)*, Fox's H- function, general class of polynomials, Beta and gamma function.*

I. Introduction

The fractional integral operator involves various special functions, which has found Significant Importance and applications in various subfield of applicable mathematical analysis. For last four decades, a number of workers like Mathai [3], Love [6], Mathai [2] , Saigo [8], etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. The Pathway Fractional integrals operator introduced by S.S. Nair [10] is defined in the following manner. Let $f(x) \epsilon L(a,b); \eta \epsilon C, R(\eta) > 0; a > 0$ and let us take a "pathway parameter" $\alpha < 1$. Then the pathway fractional integral operator is defined as follows

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_{0}^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} f(t)\,dt \tag{1}$$

The pathway model is introduced by Mathai [1] and studied further by Mathai and Haubold [3].For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p. d. f.):

$$f(x) = c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} \tag{2}$$

In case that $-\infty < x > \infty; \delta > 0; \beta \geq 0; \left[1 - a(1-\alpha)|x|^{\delta}\right] > 0; \gamma > 0$ where c is the normalizing constant and α is called the pathway parameter. For real α the normalizating constant is as follws :

$$c = \frac{1}{2}\frac{\delta\left[a(1-\alpha)\right]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha} + 1\right)}; \alpha < 1 \tag{3}$$

$$c = \frac{1}{2}\frac{\delta\left[a(1-\alpha)\right]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\beta}{1-\alpha}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha} - \frac{\gamma}{\delta}\right)}; \frac{1}{\alpha-1} - \frac{\gamma}{\delta} > 0; \alpha > 1 \tag{4}$$

$$c = \frac{1}{2}\frac{\delta\left[a\beta\right]^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}; \alpha \to 1 \tag{5}$$

Observe that for $\alpha < 1$ it is a finite range density with $\left[1 - a\left(1-\alpha\right)|x|^{\delta}\right] > 0$ and (2) remains in the extended generalized type-1 beta family. The pathway density in (3), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha > 1$, writing $1 - \alpha = -\left(\alpha - 1\right)$ we have

$$f\left(x\right) = c|x|^{\gamma-1}\left[1 + a\left(\alpha-1\right)|x|^{\delta}\right]^{\frac{-\beta}{1-\alpha}} \tag{6}$$

Provided that $-\infty < x > \infty; \delta > 0; \beta \geq 0; \gamma > 0$ which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F-density, the Student-t density, the Cauchy density and many more. Here we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (2) and (6) take the exponential form. since

$$lim_{\alpha\to1} c|x|^{\gamma-1}\left[1 - a\left(1-\alpha\right)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}}$$
$$= lim_{\alpha\to1} c|x|^{\gamma-1}\left[1 + a\left(\alpha-1\right)|x|^{\delta}\right]^{\frac{-\beta}{1-\alpha}} = c|x|^{\gamma-1}e^{a\eta|x|^{\delta}} \tag{7}$$

This includes the generalized gamma, the Weibull, the chi-square, the Laplace, Maxwell-Boltzmann and other related densities. For more details on the pathway model, the reader is referred to the recent papers of Mathai and Haubold [2], [3]. The Aleph ($\aleph$) -function introduced by Sudland [5], has the notation and complete definition is presented in the following manner which is given below in terms on the Mellin- Barnes type integrals.

$$\aleph\left[z\right] = \aleph^{m,n}_{p_i,q_i,\tau_i,r}\left[z\Big|^{(a_j,A_j)_{1,n},\ldots,[\tau_i(a_j,A_j)]_{n+1,p_i}}_{(b_j,B_j)_{1,m},\ldots,[\tau_i(b_j,B_j)]_{m+1,q_i}}\right]$$
$$= \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i,r}\left(s\right)z^{-s}ds \tag{8}$$

For all z$\neq$ 0 where $\omega = \sqrt{(-1)}$

$$\Omega^{m,n}_{p_i,q_i,\tau_i,r}\left(s\right) = \frac{\prod_{j=1}^{m}\Gamma\left(b_j + B_js\right)\prod_{j=1}^{n}\Gamma\left(1 - a_j - A_js\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\left(1 - b_{ji} - B_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji} + A_{ji}s\right)} \tag{9}$$

The integration path $L = L_{i\gamma\infty}, \gamma\epsilon R$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles , assumed to be simple of $\Gamma\left(1 - \alpha_i - A_js\right), j = 1, ..., n$ do not coincide with the pole of $\Gamma\left(\beta_i + B_js\right), j = 1, ..., m$ the parameter p_i, q_i are non negative integers satisfying $0 \leq n \leq p_i, 0 \leq m \leq q_i, \tau_i > 0$ for $i = 1, ..., r$. The $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji}\epsilon C$, The empty product in (9) is interpreted as unity. The existence conditions for the defining integral (8) are giving below

$$\phi_l > 0, |arg\left(z\right)| < \frac{\pi}{2}\phi_l, l = 1, ..., r \tag{10}$$

$$\phi_l \geq 0, |arg\left(z\right)| < \frac{\pi}{2}\phi_l and R\left(\xi_l\right) + 1 < 0 \tag{11}$$

Where

$$\phi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l \left(\sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl} \right) \tag{12}$$

$$\xi_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl} \right) + \frac{1}{2}\left(p_l - q_l\right), (i = 1, ..., r) \tag{13}$$

For details ccount of Aleph (ℵ)-function see [5],[10] and [11].

$$\aleph\left[x, y\right] = \aleph^{0,n;m_1,n_1;m_2,n_2}_{p,q;p_i,q_i,\tau_i;,p_i',q_i';r} \left[\begin{matrix} x \\ y \end{matrix} \Big|^{(a_j;\alpha_j,A_j)_{1,p};(c_j,C_j)_{1,n_1},...,[\tau_j(c_j,C_j)]_{n_1+1,p_i};(e_j,E_j)_{1,n_2},...,\left[\tau_j'(e_j,E_j)\right]_{n_2+1,p_i'}}_{(b_j;\beta_j,B_j)_{1,q};(d_j,D_j)_{1,m_1},...,[\tau_j(d_j,D_j)]_{m_1+1,q_i};(f_j,F_j)_{1,m_2},...,\left[\tau_j'(f_j,F_j)\right]_{m_2+1,q_i'}} \right]$$

$$= \frac{1}{(2\pi\omega)^2} \int_L \int_L \phi\left(s, \xi\right) \varphi_1\left(s\right) \varphi_2\left(\xi\right) x^{-s} y^{-\xi} ds d\xi \tag{14}$$

where $\omega = \sqrt{(-1)}$

$$\phi\left(s, \xi\right) = \frac{\prod_{j=1}^{n} \Gamma\left(1 - a_j - \alpha_j s - A_j \xi\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_j + \alpha_j s + A_j \xi\right) \prod_{j=1}^{q} \Gamma\left(1 - b_j - \beta_j - B_j \xi\right)} \tag{15}$$

$$\varphi_1\left(s\right) = \Omega^{m_1,n_1}_{p_i,q_i,\tau_i,r}\left(s\right) = \frac{\prod_{j=1}^{m_1} \Gamma\left(d_j + D_j s\right) \prod_{j=1}^{n_1} \Gamma\left(1 - c_j - C_j s\right)}{\sum_{i=1}^{r} \tau_i \prod_{j=m_1+1}^{q_i} \Gamma\left(1 - d_{ji} - D_{ji} s\right) \prod_{j=n_1+1}^{p_i} \Gamma\left(c_{ji} + C_{ji} s\right)} \tag{16}$$

$$\varphi_2\left(\xi\right) = \Omega^{m_2,n_2}_{p_i',q_i',\tau_i',r}\left(\xi\right) = \frac{\prod_{j=1}^{m_2} \Gamma\left(f_j + F_j \xi\right) \prod_{j=1}^{n_2} \Gamma\left(1 - e_j - E_j \xi\right)}{\sum_{i=1}^{r} \tau_i' \prod_{j=m_2+1}^{q_i'} \Gamma\left(1 - f_{ji} - F_{ji} \xi\right) \prod_{j=n_2+1}^{p_i'} \Gamma\left(e_{ji} + E_{ji} \xi\right)} \tag{17}$$

Theorem:

With the set of sufficient conditions (10), (11), (12) and (13), let $\eta, \beta, \gamma, \rho \epsilon C, R\left(\beta\right), R\left(\eta\right), R\left(\gamma\right), R\left(\rho\right) > 0, R\left(\delta\right) > 0, Re\left(1 + \frac{\eta}{(1-\alpha)}\right) > 0$,

$$P^{(\eta,\alpha)}_{0+}\left[\left(t^{\rho-1} \aleph^{m_1,n_1}_{p_i,q_i,\tau_i,r} \left[\lambda t^{\mu} \Big|^{(a_j,A_j)_{1,n_1},...,[\tau_i(a_j,A_j)]_{n_1+1,p_i}}_{(b_j,B_j)_{1,m_1},...,[\tau_i(b_j,B_j)]_{m_1+1,q_i}} \right] \right) \times \right.$$
$$\left. \left\{ \aleph^{m_2,n_2}_{p_i',q_i',\tau_i',r} \left[\omega t^{\nu} \Big|^{(c_j,C_j)_{1,n_2},...,[\tau_i(c_j,C_j)]_{n_2+1,p_i'}}_{(d_j,D_j)_{1,m_2},...,[\tau_i(d_j,D_j)]_{m_2+1,q_i'}} \right] \right\} \right]$$
$$= \frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}} \Gamma\left(\frac{\eta}{1-\alpha}\right)$$
$$\aleph^{0,1;m_1,n_1;m_2,n_2}_{1,1;p_i,q_i,\tau_i;p_i',q_i',\tau_i';r} \left[\begin{matrix} \frac{\lambda x^{\mu}}{a(1-\alpha)^{\mu}} \\ \frac{\omega x^{\nu}}{a(1-\alpha)^{\nu}} \end{matrix} \Big|^{(1-\sigma,\mu,\nu);(a_j,A_j)_{1,n_1},...,[\tau_i(a_j,A_j)]_{n_1+1,p_i};(c_j,C_j)_{1,n_2},...,\left[\tau_i'(c_j,C_j)\right]_{n_2+1,p_i'}}_{\left(-\frac{\eta}{1-\alpha}-\sigma,\mu,\nu\right);(b_j,B_j)_{1,m_1},...,[\tau_i(b_j,B_j)]_{m_1+1,q_i},(d_j,D_j)_{1,m_2},...,\left[\tau_i'(d_j,D_j)\right]_{m_2+1,q_i'}} \right] \tag{18}$$

Proof:

Using the definitions (1), (8), (14) then by interchange the order of integrations and summations (which is permissible under the conditions stated above), evaluate inner integral by making use of beta and gamma function formula, we arrive at the desired results.

Special Cases :

1. If we take $\tau_i = 1, \tau_i' = 1, i = 1, 2, ..., r$ in (18) then we reduce the following results in term of I- function.

$$P^{(\eta,\alpha)}_{0+}\left[\left(t^{\rho-1} I^{m_1,n_1}_{p_i,q_i,1,r} \left[\lambda t^{\mu} \Big|^{(a_j,A_j)_{1,n_1},...,[(a_j,A_j)]_{n_1+1,p_i}}_{(b_j,B_j)_{1,m_1},...,[(b_j,B_j)]_{m_1+1,q_i}} \right] \right) \times \right.$$
$$\left. \left\{ I^{m_2,n_2}_{p_i',q_i',1,r} \left[\omega t^{\nu} \Big|^{(c_j,C_j)_{1,n_2},...,[(c_j,C_j)]_{n_2+1,p_i'}}_{(d_j,D_j)_{1,m_2},...,[(d_j,D_j)]_{m_2+1,q_i'}} \right] \right\} \right]$$
$$= \frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}} \Gamma\left(\frac{\eta}{1-\alpha}\right)$$

$$I^{0,1;m_1,n_1;m_2,n_2}_{1,1;p_i,q_i;p_i',q_i';r}\left[\begin{array}{c}\frac{\lambda x^{\mu}}{a(1-\alpha)^{\mu}}\\ \frac{\omega x^{\nu}}{a(1-\alpha)^{\nu}}\end{array}\Big|\begin{array}{l}(1-\sigma,\mu,\nu);(a_j,A_j)_{1,n_1},\dots,[(a_j,A_j)]_{n_1+1,p_i};(c_j,C_j)_{1,n_2},\dots,[(c_j,C_j)]_{n_2+1,p_i'}\\ \left(-\frac{\eta}{1-\alpha}-\sigma,\mu,\nu\right);(b_j,B_j)_{1,m_1},\dots,[(b_j,B_j)]_{m_1+1,q_i},(d_j,D_j)_{1,m_2},\dots,[(d_j,D_j)]_{m_2+1,q_i'}\end{array}\right] \quad (19)$$

2. If we Substitute $\tau_i = 1, \tau_i' = 1, i = 1, 2, ..., r$ and $r = 1$ in (18) then we get the following results reduce in terms of H- funtion [11]

$$\left\{P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}H^{m_1,n_1}_{p_i,q_i,1,1}\left[\lambda t^{\mu}\Big|^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}\right]H^{m_2,n_2}_{p_i',q_i',1,1}\left[\omega t^{\nu}\Big|^{(c_j,C_j)_{1,p'}}_{(d_j,D_j)_{1,q'}}\right]\right)\right\}$$
$$= \frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}}\Gamma\left(\frac{\eta}{1-\alpha}\right)$$
$$H^{0,1;m_1,n_1;m_2,n_2}_{1,1;p_i,q_i;p_i',q_i'}\left[\begin{array}{c}\frac{\lambda x^{\mu}}{a(1-\alpha)^{\mu}}\\ \frac{\omega x^{\nu}}{a(1-\alpha)^{\nu}}\end{array}\Big|\begin{array}{l}(1-\sigma,\mu,\nu);(a_j,A_j)_{1,p};(c_j,C_j)_{1,p'}\\ \left(-\frac{\eta}{1-\alpha}-\sigma,\mu,\nu\right);(b_j,B_j)_{1,q},(d_j,D_j)_{1,q'}\end{array}\right] \quad (20)$$

3. . If we use the relation with Mittage- Leffler function, obtained [4] given as $E^{\delta}_{\beta,\gamma} = \frac{1}{\Gamma(\delta)}H^{1,1}_{1,2}\left[-z\Big|^{(1-\delta,1)}_{(0,1),(1-\gamma,\beta)}\right]$ in (18) , then we arrive at

$$\left\{P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}E^{\delta_1}_{\xi_1,\eta_1}\left[\lambda t^{\mu}\right]E^{\delta_2}_{\xi_2,\eta_2}\left[\lambda t^{\nu}\right]\right)\right\}$$
$$= \frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}}\Gamma\left(\frac{\eta}{1-\alpha}\right)$$
$$H^{0,1;1,1;1,1}_{1,1;1,2;1,2}\left[\begin{array}{c}\frac{\lambda x^{\mu}}{a(1-\alpha)^{\mu}}\\ \frac{\omega x^{\nu}}{a(1-\alpha)^{\nu}}\end{array}\Big|\begin{array}{l}(1-\sigma,\mu,\nu);(1-\delta_1,\xi_1);(1-\delta_2,\xi_2)\\ \left(-\frac{\eta}{1-\alpha}-\sigma,\mu,\nu\right);(0,1);(1-\eta_1,\xi_1),(0,1);(1-\eta_2,\xi_2)\end{array}\right] \quad (21)$$

References

[1]. A.M. Mathai, 2005. A pathway to matrix-variate gamma and normal densities. Linear Algebra Appl. 396, 317-328.
[2]. A.M. Mathai and H. J. Haubold , 2008. On generalized distributions and path-ways. Phys. Lett. A 372, 2109-2113.
[3]. A.M. Mathai and H. J.Haubold, 2007. Pathway model, super statistics, Tsallis statistics and a Generalize measure of entropy. Phys. A 375, 110-122.
[4]. V.B.L. Chaurasia and V. Gill, 2013.New pathway fractional integral operator involving H -Functions, ISSN: 2090-5858, Vol. 4(1), 160-168.
[5]. B.Sudland , B. Baumann and T.F.Nonnenmacher , 2001. Fractional driftless, Fokker-Planck Equation with power law delusion coecient in V.G. Ganga, E.W. Mayr, W.G. Varozhtsov, editors Computer Algebra in Scienti c computing (CASC Konstanz2001) Springer, Berlin, 513-525.
[6]. E. R. Love,1967.Some integral equations involving hyper geometric functions, Proc. Edin. Math. Soc 15(3), 169-198.
[7]. H.M. Shrivastava, 1972. A contour integral involving Fox`s H-function Indian J.Math.Vol 14, 1-6.
[8]. M.Saigo, 1978. A remark on integral operators involving the Gauss hyper geometric functions, Math. Rep. Kyushu Univ. 11135-143.
[9]. N.Sudland, B.Bauman and T.F.Nonnenmacher, 1998. Open problem, who know about the Aleph functions? Fract. Calc. Appli. Anal.1 (4), 401-402.
[10]. S. S. Nair, 2009. Pathway fractional integration operator, Fract. Calc. Appl. Anal. 12(3), 237-252.
[11]. R.K.Saxena, , J.Ram, and D.kumar, 2013. Generalized fractional calculus of the product of two Aleph functions associated with the Appell function , Kournal of Mathematics, Volume 45(2), 137-148.