IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 2 Ver. V (Mar. - Apr. 2017), PP 40-46
www.iosrjournals.org

Existence of Necessary Condition for Normal Solution Operator Equation

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Abstract: *An operator means a bounded linear operator on Hilbert span. This paper proves the assertion made in its title. Following theorem yields the famous result*

$$AB + BA^* = I = A^*B + BA \qquad (1)$$

Where A and B are the bounded linear operator on a Hilbert span H. where B^* is self adjoint satisfying the above equation. After modification of this equation some interesting results are obtained.

Theorem: If $AB + BA^* = I$ has solution B, then $0 \in \sigma_\delta$ (A) and $0 \in \rho(B)$. Further $\|B^{-1}\| \le 2\|A\|$. and $0 \in \rho$ (Re B).

Proof. If σ_π (A) denotes the approximate point spectrum of A then $\sigma_\pi(A^*) = \overline{\sigma_\delta(A)}$ bar is complex conjugate. Let $0 \in \sigma_\delta(A)$, then $0 \in \sigma_\pi(A^*)$, and so there exist a sequence $\{X_n\}$ of unit vector in H such that $A^*x_n \to 0$ as $n \to \infty$.

Now $AB + BA^* = I$

$$\Rightarrow \quad 1 = (x_n, x_n) = ((AB + BA^*)x_n, x_n)$$

or, $$1 = (ABx_n, x_n) + (BA^*x_n, x_n)$$

$$= \quad (Bx_n, A^*x_n) + (A^*x_n, Bx_n) \to 0$$

as $n \to \infty$.

This is contradiction, hence $0 \notin \sigma_\delta(A)$.

Similarly we can prove that $0 \notin \sigma_\delta(B)$. since B is a self adjoint, we have $\sigma_\delta(B) = \sigma(B)$ Hence $0 \in \rho(B)$

Next we suppose that $x \in$ H be arbitrary then

$$\|x\|^2 = ((AB + BA^*)x, x)$$

$$= (ABx, x) + (BA^*x, x)$$

$$= (Bx, A^*x) + (A^*x, Bx)$$

$$\le \|Bx\|\|A^*x\| + \|A^*x\|\|Bx\|$$

$$= 2\|Bx\|\|Ax\|$$

i.e. $$1 \le 2\|B\|\|A\|$$

i.e. $\left\|B^{-1}\right\| \le 2\|A\|$

Now we suppose that $0 \in \sigma_\delta$ (Re A), this means that $0 \in \sigma_\pi$ (Re A), hence there exists a sequence $\{x_n\}$ of unit vectors in H such that $\mathrm{Re}(A)x_n \to 0$ as $n \to \infty$
we have

$$2I = AB + BA^* + A^*B + BA$$

$$\Rightarrow \quad 2(x_n, x_n) = \left(\left(A + A^*\right)B + B\left(A + A^*\right)x_n, x_n\right)$$

or, $$2 = 2\left[\left(\mathrm{Re}\, ABx_n, x_n\right) + \left(B\,\mathrm{Re}\, Ax_n, x_n\right)\right]$$

or, $$I = \left(Bx_n, \mathrm{Re}\, Ax_n\right) + \left(\mathrm{Re}\, Ax_n, Bx_n\right) \to 0$$

as $n \to \infty$

This is contradiction, hence $0 \notin \sigma_\delta$ (Re A). Since (Re A) is self adjoint, hence σ_δ (Re A)= σ (Re A).

$$\Rightarrow \quad 0 \in \rho(\mathrm{Re}\, A)$$

Theorem : If (*i*) has a solution *B* > 0 (*i.e.* (*Bx*, *x*) > 0 for all $x \in H/\{0\}$). Then there is an inner product on *H*, equivalent to the inner product (.,.), such that *Re A* > 0 with respect to it.

Proof. We define a new equivalent inner product on H by

$$\langle x, y\rangle = (x, By), \text{for all}\, x, y \in H$$

Since $\left[B, A - A^*\right] = 0$ we have

$$B\left(A - A^*\right) = \left(A - A^*\right)B$$

$$\Rightarrow \quad \left(B\left(A - A^*\right)x, x\right) = \left(\left(A - A^*\right)Bx, x\right)$$

or, $$\left(\left(A - A^*\right)x, Bx\right) = \left(Bx, \left(A^* - A\right)x\right)$$

$$= \overline{\left(\left(A^* - A\right)x, Bx\right)}$$

or, $$\left\langle\left(A - A^*\right)x, x\right\rangle = \left\langle\overline{\left(A^* - A\right)x, x}\right\rangle$$

or, $$\langle Ax, x\rangle - \langle A^*x, x\rangle = \overline{\langle A^*x, x\rangle} - \overline{\langle Ax, x\rangle}$$

or, $$\langle Ax, x\rangle + \overline{\langle Ax, x\rangle} = \langle A^*x, x\rangle + \overline{\langle A^*x, x\rangle}$$

or, $$\mathrm{Re}\langle Ax, x\rangle = \mathrm{Re}\langle A^*x, x\rangle \qquad (2)$$

Again from the equation (1) we have

$$2I = \left(A + A^*\right)B + B\left(A + A^*\right)$$ then for any $x \in H.$

$$2(x, x) = \left(\left(A + A^*\right)Bx, x\right) + \left(B\left(A + A^*\right)x, x\right)$$

Or, $$2\|x\|^2 = \left(Bx, \left(A^* + A\right)x\right) + \left(\left(A + A^*\right)x, Bx\right)$$

$$= \overline{\left(\left(A^* + A\right)x, Bx\right)} + \left(\left(A + A^*\right)x, Bx\right)$$

$$= \overline{\left\langle\left(A + A^*\right)x, x\right\rangle} + \left\langle\left(A + A^*\right)x, x\right\rangle$$

$$= \left\langle \overline{Ax+x} \right\rangle + \left\langle A^* x, x \right\rangle + \left\langle Ax, x \right\rangle + \left\langle A^* x, x \right\rangle$$

$$= \left\langle Ax, x \right\rangle + \left\langle \overline{Ax, x} \right\rangle + \left\langle A^* x, x \right\rangle + \left\langle \overline{A^* x + x} \right\rangle$$

$$= 2\operatorname{Re}\left\langle Ax, x \right\rangle + 2\operatorname{Re}\left\langle A^* x, x \right\rangle$$

$$= 2\operatorname{Re}\left\langle Ax, x \right\rangle + 2\operatorname{Re}\left\langle Ax, x \right\rangle \qquad [\text{By}(2)]$$

$$= 4\operatorname{Re}\left\langle Ax, x \right\rangle$$

Or, $\operatorname{Re}\left\langle Ax, x \right\rangle = \frac{1}{2} \| x \|^2 > 0$ for all $x \in H$

Hence $\operatorname{Re} A > 0$

Theorem : If there exists a solutions A to (1) such that the eigen vectors of A^* span *H*, then $W(B) \subseteq R/\{0\}$.

Proof. Let $\lambda \in \sigma_P\left(A^*\right)$ (* the point spectrum of A^*) and let $x \in H, x \notin 0$ be an eigen vector corresponding to λ. So we have $A^* x = \lambda x$ for all $x \in H$.

From (1) we have

$$0 = \left(\left(A - A^*\right)Bx, x\right) + \left(B\left(A^* - A\right)x, x\right)$$

$$or, \quad 0 = \left(Bx, \left(A^* - A\right)x\right) + \left(\left(A^* - A\right)x, Bx\right)$$

$$or, \quad 0 = \left(Bx, \lambda x - Ax\right) + \left(\lambda x - Ax, Bx\right)$$

$$or, \quad 0 = \left(Bx, \lambda x\right) - \left(Bx, Ax\right) + \left(\lambda x, Bx\right) - \left(Ax, Bx\right)$$

$$or, \quad 0 = \overline{\lambda}\left(Bx, x\right) - \left(Bx, Ax\right) + \lambda\left(Bx, x\right) - \left(Ax, Bx\right)$$

$$or, \quad 0 = \left(Bx, Ax\right) + \left(\overline{Bx, Ax}\right) + \left(\lambda + \overline{\lambda}\right)\left(Bx, x\right)$$

Or, $\operatorname{Re}\left(Bx, Ax\right) = \left(\operatorname{Re}\lambda\right)\left(Bx, x\right)$ (3)

Again we have

$$2\| x \|^2 = \left(\left(A + A^*\right)Bx, x\right) + \left(B\left(A^* + A\right)x, x\right)$$

$$= \left(Bx, \left(A^* + A\right)x\right) + \left(\left(A^* + A\right)x, Bx\right)$$

$$= \left(Bx, \lambda x + Ax\right) + \left(\lambda x + Ax, Bx\right)$$

$$= \left(Bx, \lambda x\right) + \left(Bx, Ax\right) + \left(\lambda x, Bx\right) + \left(Ax, Bx\right)$$

$$= \overline{\lambda}\left(Bx, x\right) + \lambda\left(Bx, x\right) + \left(Bx, Ax\right) + \left(\overline{Bx, Ax}\right)$$

$$= \left(\lambda + \overline{\lambda}\right)\left(Bx, x\right) + 2\operatorname{Re}\left(Bx, Ax\right)$$

$$= 2\left[\operatorname{Re}\lambda\left(Bx, x\right) + \operatorname{Re}\lambda\left(Bx, x\right)\right] \qquad by(3)$$

$$or, \quad 2\|x\|^2 = 4(\operatorname{Re}\lambda)(Bx,x)$$

$$or, \quad 1 = 2(\operatorname{Re}\lambda)(By,y) \quad \text{where } y = \frac{x}{\|x\|}$$

So that $\|y\| = 1.$ Clearly $\operatorname{Re}\lambda \neq 0$

This implies that

$$(By,y) = \frac{1}{2(\operatorname{Re}\lambda)} \text{ for all } y \in H$$

Hence $W(B) \subseteq R/\{0\}.$

Theorem: If there exists a solution A to (1) such that the eigen vectors of A span H, then $W(B) \subseteq R/\{0\}.$

Proof. Let $\lambda \in \sigma_P(A)$ and let $0 \notin x \in H$ be an eigen vector corresponding to λ. Then we have *Ax* = *x* for all $x \in H$

From equation (1) we have

$$\begin{aligned}
0 &= ((A-A^*)Bx,x) + (B(A^*-A)x,x) \\
&= (Bx,(A^*-A)x) + ((A^*-A)x,Bx) \\
&= (Bx,A^*x-\lambda x) + (A^*x-\lambda x,Bx) \\
&= (Bx,A^*x) - \overline{\lambda}(Bx,x) + (A^*x,Bx) - \lambda(x,Bx) \\
&= (Bx,A^*x) + (\overline{Bx,A^*x}) + \overline{\lambda}(Bx,x) - \lambda(Bx,x)
\end{aligned}$$

$$or, \quad (\lambda+\overline{\lambda})(Bx,x) = (Bx,A^*x) + (\overline{Bx,A^*x})$$

Or, $$\operatorname{Re}\lambda(Bx,x) = \operatorname{Re}(Bx,A^*x) \tag{4}$$

Again we have

$$\begin{aligned}
or, \quad 2\|x\|^2 &= ((A+A^*)Bx,x) + (B(A^*+A)x,x) \\
or, \quad 2\|x\|^2 &= (Bx,(A^*+A)x) + ((A^*+A)x,Bx) \\
&= (Bx,A^*x+\lambda x) + (A^*x+\lambda x,Bx) \\
&= (Bx,A^*x) + \overline{\lambda}(Bx,x) + (A^*x,Bx) + \lambda(x,Bx) \\
&= (\lambda+\overline{\lambda})(Bx,x) + (Bx,A^*x) + (\overline{Bx,A^*x}) \\
&= 2\operatorname{Re}\lambda(Bx,x) + 2\operatorname{Re}(Bx,A^*x) \\
or, \quad 2\|x\|^2 &= 2[\operatorname{Re}\lambda(Bx,x) + \operatorname{Re}\lambda(Bx,x)] \qquad by(4)
\end{aligned}$$

$$or, \quad 2\|x\|^2 = 4\operatorname{Re}\lambda(Bx,x)$$

$$or, \quad 1 = 2\operatorname{Re}\lambda(By,y) \quad \text{where } y=\frac{x}{\|x\|}$$

$$\Rightarrow \quad (By,y)= \frac{1}{2\operatorname{Re}\lambda},(\operatorname{Re}\lambda\neq 0)$$

This show that $W(B)\subseteq R/\{0\}$.

Theorem : If there exists a solution B to equation (1) such that the eigen vectors of B span H, then either $W(A)\subseteq R/\{0\} or W(\operatorname{Re}A)\subseteq R/\{0\}$.

Proof. Let $\lambda\in\sigma_P(B)$ and let $0\neq x\in H$ be an eigen vector corresponding to λ. Then $Bx=\lambda x$ for all $x\in H$.

From equation is (1) we have

$$\begin{aligned}
0 &= \left(\left(A-A^*\right)Bx,x\right)+\left(B\left(A^*-A\right)x,x\right)\\
&= \left(Bx,\left(A^*-A\right)x\right)+\left(\left(A^*-A\right)x,Bx\right)\\
&= \left(\lambda x,\left(A^*-A\right)x\right)+\left(\left(A^*-A\right)x,\lambda x\right)\\
&= \lambda\left[\left(x,A^*x\right)-\left(x,Ax\right)\right]+\bar{\lambda}\left[\left(A^*x,x\right)-\left(Ax,x\right)\right]\\
&= \lambda\left[\left(Ax,x\right)-\left(\overline{Ax,x}\right)\right]-\bar{\lambda}\left[\left(Ax,x\right)-\left(\overline{Ax,x}\right)\right]\\
&= \left(\lambda-\bar{\lambda}\right)\left[\left(Ax,x\right)-\left(\overline{Ax,x}\right)\right]\\
&= \left(2\mathrm{I_m}\lambda\right)\left(2\mathrm{I_m}\left(Ax,x\right)\right)\\
&= 4\left(\mathrm{I_m}\lambda\right)\left(\mathrm{I_m}\left(Ax,x\right)\right)\\
&\Rightarrow \text{Either } \mathrm{I_m}\left(Ax,x\right)=0
\end{aligned}$$

In this case $(Ax,x)=\left(\overline{Ax,x}\right)$ (5)

Or $\mathrm{I_m}\lambda=0 \text{ In this case } \lambda=\bar{\lambda}$ (6)

Again from (1) we have

$$\begin{aligned}
2\|x\|^2 &= \left(\left(A+A^*\right)Bx,x\right)+\left(B\left(A+A^*\right)x,x\right)\\
&= \left(Bx,\left(A^*+A\right)x\right)+\left(\left(A^*+A\right)x,Bx\right)\\
&= \left(\lambda x,\left(A^*+A\right)x\right)+\left(\left(A^*+A\right)x,\lambda x\right)\\
&= \lambda\left[\left(x,A^*x\right)+\left(x,Ax\right)\right]+\bar{\lambda}\left[\left(A^*x,x\right)+\left(Ax,x\right)\right]\\
&= \lambda\left[\left(Ax,x\right)+\left(\overline{Ax+x}\right)\right]+\bar{\lambda}\left[\left(Ax,x\right)+\left(\overline{Ax+x}\right)\right]
\end{aligned}$$

$$= \quad \left(\lambda+\overline{\lambda}\right)\left[\left(Ax,x\right)+\left(\overline{Ax,x}\right)\right] \tag{7}$$

Now two cases arise

Case I : When $\left(Ax,x\right)=\left(\overline{Ax,x}\right)$ then by (7)

We have

$$2\|x\|^2 = 4\left(\operatorname{Re}\lambda\right)\left(Ax,x\right)$$

$$\Rightarrow \quad l = 2\left(\operatorname{Re}\lambda\right)\left(Ay,y\right) \quad \text{where} \quad y=\frac{x}{\|x\|}$$

Obviously by $\operatorname{Re}\lambda \neq 0.$ This implies that

$$\left(Ay,y\right)=\frac{1}{2\left(\operatorname{Re}\lambda\right)} \text{ for all } y\in H.$$

Hence $W\left(A\right)\subseteq R/\{0\}.$

Case II : When $\lambda=\overline{\lambda}$; *i.e.* λ is purely real from equation (7) we have

$$2\|x\|^2 = 2\lambda\left[\left(Ax,x\right)+\left(\overline{Ax,x}\right)\right]$$

$$= 2\lambda\, 2\operatorname{Re}\left(Ax,x\right)$$

$$= 4\lambda\operatorname{Re}\left(Ax,x\right)$$

$$\Rightarrow \quad \|x\|^2 = 2\lambda\operatorname{Re}\left(Ax,x\right)$$

$$\Rightarrow \quad 1 = 2\lambda\operatorname{Re}\left(Ay,y\right) \text{ where } y=\frac{x}{\|x\|}$$

$$= \operatorname{Re}\left(Ay,y\right)=\frac{1}{2\lambda} \text{ for all } y\in H \tag{8}$$

We have

$$2a = z+\overline{z} \quad i.e. \quad a=\frac{1}{2}\left(z+\overline{z}\right)$$

i.e. $\operatorname{Re}\left(z\right) = \frac{1}{2}\left(z+\overline{z}\right)$

$$= \frac{1}{2}\left((Ay,y)+\left(\overline{Ay,y}\right)\right)$$

$$= \frac{1}{2}\left((Ay,y)+\left(\overline{y,A^{*}y}\right)\right)$$

$$= \frac{1}{2}\left((Ay,y)+\left(A^{*}y,y\right)\right)$$

$$= \frac{1}{2}\left(\left(A+A^{*}\right)y,y\right)$$

$$= \left(\frac{1}{2}\left(A+A^{*}\right)y,y\right)$$

$$= \left(\mathrm{Re}(A)y,y\right) \qquad (9)$$

Hence from (8) and (9) we have

$$\left((\mathrm{Re}\,A)y,y\right) = \frac{1}{2\lambda}\text{ for all } y\in H$$

$$\Rightarrow \quad W(\mathrm{Re}\,A)\subseteq R/\{0\}$$

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