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Bivariate Beta Exponential Distributions

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Abstract: *The exponential distribution is perhaps the most widely applied statistical distribution in reliability. Anew continuous bivariate distribution called the bivariate beta-exponential distribution (BBE) that extends the bivariate exponential distribution are proposed. We introduce a new bivariate beta-exponential distributions (BBE) based on some types of copulas. Parametric and semiparametric methods are used to estimate the parameters of the models. Finally, Simulation is studied to illustrate methods of inference discussed and examine the satisfactory performance of the proposed distributions.*

Key words*: Beta exponential distribution, beta G distribution, bivariate beta exponential distributions; Maximum likelihood method; copula;*

I. Introduction

The exponential distribution is a popular distribution the most widely used and applied for analyzing lifetime data and for problems in reliability.

The exponential distribution is a popular distribution widely used for analyzing lifetime data. The exponential distribution is perhaps the most widely applied statistical distribution for problems in reliability. In this aim, we consider a generalization referred to as the beta exponential distribution generated from the logit of a beta random variable. We work with the beta exponential (BE) distribution because of the wide applicability of the exponential distribution and the fact that it extends some recent developed distributions. An application is illustrated to a real data set with the hope that it will attract more applications in reliability, biology and other areas of research.

Eugene et al. (2002) introduced the beta distribution as a generator to suggest the so-called family of beta G distributions. The cumulative distribution function (c.d.f.) of a beta-G random variable X is defined as

$$\mathrm{F}(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)}\int_0^{G(x)} w^{a-1}(1-w)^{b-1}dw, \qquad (1)$$

for G(x) is the cdf of any random variable, a > 0 and b > 0, where $I_y(a,b) = B_y(a,b)/B(a,b)$denotes the incomplete beta function ratio, and $B_y(a,b) = \int_0^y w^{a-1}(1-w)^{b-1}dw$ denotes the incomplete beta function.

The p.d.f. corresponding to the beta-G distribution in (1) is given by

$$f(x) = \frac{1}{B(a,b)}G^{a-1}(x)\big(1-G(x)\big)^{b-1}g(x). \qquad (2)$$

where g(x) = dG(x)/dx is the pdf of the parent distribution. The pdf f(x) will be most tractable when the functions G(x) and g(x) have simple analytic expressions.

This family of distributions is a generalization of the distributions of order statistics for the random variable X with cdf F(x) as pointed out by Eugene, et al. (2002) and Jones (2004). Since the paper by Eugene et al. (2002), many beta-G distributions have been studied in the literature including the beta-Gumbel distribution by Nadarajah and Kotz (2004), beta exponential distribution by Nadarajah and Kotz (2006), beta-Weibull distribution by Famoye et al.(2005) and Cordeiro et al., (2011).

For more details, see, also, the beta-Pareto distribution by Akinsete, et al. (2008), beta modified Weibull distribution by Silva et al.(2010), beta generalized half-normal distribution by Pescim et al., (2010), And ,also, the beta Burr XII distribution by Paranaiba, et al., (2011), beta extended Weibull distribution by Cordeiro, et al.(2012),beta exponentiated Weibull by Cordeiro et al.(2013),beta-lindley distribution by Merovci and Sharma (2014), beta Burr type X distribution by Merovci et al.,(2016).

Eugene et al. (2002) introduced the beta normal distribution by taking G(x) in (1) to be the cdf of the normal distribution. Nadarajah and Kotz (2004) introduced the beta Gumbel (BG) distribution by taking G(x) to be the cdf of the Gumbel distribution. Also, Nadarajah and Kotz (2006) studied the BE distribution and obtained the moment generating function, the asymptotic distribution of the extreme order statistics and discussed the maximum likelihood estimation. For more details, see Azzalini(1985), Alexander, et al., (2012),Nadarajah and Rocha (2016a), Nadarajah and Rocha (2016b), Alzaatreh et al., (2013), Aljarrah et al., (2014), Nadarajah, et al., (2015).

Beta exponential distribution used effectively in different lifetime applications. Nadarajah and Kotz (2006) first introduced it.

We now study the BE distribution by taking G(x) in (1) to be the cdf of the exponential (E) distribution. Then, the beta exponential (BE) distribution with three parameters α > 0, a > 0 and b >0 with the following Cdf and the pdf, respectively,

$$F(x) = \frac{1}{B(a,b)} \int_0^{1-exp(-\alpha x)} w^{a-1}(1-w)^{b-1} dw$$

The simple formula for the cdf of BE distribution if a, bare real integer given by

$$F(x) = \frac{[1-\exp(-\alpha x)]^a}{\Gamma(a)} \sum_{j=0}^{b-1} \frac{\Gamma(a+j)}{j!} \{\exp(-j\alpha x)\}, \qquad (3)$$

And, the pdf given by

$$f(x) = \frac{\alpha}{B(a,b)} [1-\exp(-\alpha x)]^{a-1} \exp(-b\alpha x), x > 0, \alpha, a, b > 0. \qquad (4)$$

And the hazard rate function given by

$$h(x) = \frac{\alpha \exp(-b\alpha x)[1-\exp(-\alpha x)]^{a-1}}{B(a,b) I_{\exp(-\alpha x)}(a,b)} . \qquad (5)$$

The moment generating function (mgf) of the BE distribution if b>2 is integer is given by

$$M(t) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \binom{b-1}{j} (-1)^j B(1-t/\alpha, a+j).$$

The simple formula for the mgf of BE distribution is given by

$$M(t) = \frac{B(b-t/\alpha, a)}{B(a,b)}. \qquad (6)$$

The r th moment of the BE distribution if b is integer can be obtain from

$$\acute{\mu}_r = \frac{\Gamma(b)}{\alpha^r B(a,b)} \sum_{j=0}^{b-1} \binom{b-1}{j} (-1)^{j+r} \left| \frac{d^r B(p,(a+j))}{dp^r} \right|_{p=1} . \qquad (7)$$

The first four moments of the BE distribution if b>0 is integer are obtain, respectively,

$$\acute{\mu}_1 = \frac{\Gamma(a+b)}{\alpha\Gamma(a)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^j}{a+j} c_j,$$

$$\acute{\mu}_2 = \frac{\Gamma(a+b)}{\alpha^2\Gamma(a)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^j}{a+j} d_j,$$

$$\acute{\mu}_3 = \frac{\Gamma(a+b)}{\alpha^3\Gamma(a)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^j}{a+j} e_j,$$

$$\acute{\mu}_4 = \frac{\Gamma(a+b)}{\alpha^4\Gamma(a)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^j}{a+j} f_j,$$

Where $c_j, d_j, e_j and\ f_j\ are\ given\ by$

$$c_j = \psi(a+j+1) + \psi(1),$$

$$d_j = c_j^2 + \acute{\psi}(1) - \psi(a+j+1) + \psi(1),$$

$$e_j = -c_j[c_j^2 + 3\{\acute{\psi}(1) - \acute{\psi}(a+j+1)\}] + \acute{\acute{\psi}}(1) - \acute{\acute{\psi}}(a+j+1),$$

$$f_j = \{c_j^2 + \acute{\psi}(1) - \acute{\psi}(a+j+1)\}[c_j^2 + 3\{\acute{\psi}(1) - \acute{\psi}(a+j+1)\}] + 2c_j^2\{\acute{\psi}(1) - \acute{\psi}(a+j+1)\} - 4c_j\{\acute{\acute{\psi}}(1) - \acute{\acute{\psi}}(a+j+1)\},$$

The BE distribution contains as special cases three well-known distributions. For example, it simplifies to the BW distribution when. If α = 1, the BE distribution becomes the beta standard exponential (BSE) distribution, If b = 1, the BE distribution becomes the EE distribution, The Exponential distribution is clearly a special case for a = b = 1.

Copulas are a general tool to construct multivariate distributions and to find dependence structure between random variables. However, the concept of copula is popular in multivariate analyses. In this aim, we show that copulas can be important used to solve many statistical problems. Stated that any multivariate distribution can be disintegrated to a copula and its continues marginal.

The Gaussian copula gives the following form

$$C(u,v) = \varphi_\Sigma(\varphi^{-1}(u), \varphi^{-1}(v)), \qquad (8)$$

where φ_Σ denotes the distribution function of a bivariate standard normal random variable and φ^{-1}represents its inverse.

The Farlie-Gumbel-Morgensten copula (FGM) takes the following form

$$C(u,v) = \mathrm{uv}[1 + \theta(1-\mathrm{u})(1-\mathrm{v})], \qquad (9)$$

where u and v ∈ I, and θ ∈ $[-1,1]$ is a dependence parameter.

Although the FGM copula family is tractable mathematically, it does not model high dependences. The range of the dependence measures Kendall's tau τ and Spearman's rho ρ are $\tau \in [-0.222, 0.222]$ and $\rho \in [-0.333, 0.333]$ respectively.

The Plackett copula takes the following form

$$\mathrm{C(u,v)} = \frac{1+(\theta-1)(u+v)}{2(\theta-1)} - \frac{\sqrt{[1+(\theta-1)(u+v)]^2 - 4uv\theta(\theta-1)}}{2(\theta-1)}, \tag{10}$$

Where $\theta \in (0, \infty)$. The correlation measure Spearman's rho is $\rho = \frac{\theta+1}{\theta-1} - \frac{2\theta\log(\theta)}{(\theta-1)^2}$. There is no closed expression in θ for the correlation measure Kendall's tau.

Several multivariate and bivariate lifetime distributions are derived using copula functions such as Johnson, et al.(1992), Nelsen(1999),Adham and Walker, (2001),Trivedi and Zimmer, (2005),Adham, et al. (2009), Kundu, et al.(2009), Kunduet al. (2010), Kundu and Gupta, (2011), Ateya and Al-Alazwari, (2013), Sarabia et al. (2014), Abd Elaal et al.(2016), Adham et al. (2016), and Abd Elaal et al.(2017).

The main article of this article is to introduce bivariate beta exponential (BBE) models based on most used copula functions in the literature as the Gaussian, Frank, Clayton, and Farlie-Gumbel-Morgensten (FGM) and suggest which of them is more suitable. In addition, the performance of the proposed BBE will be examined using a real data example.

The contents of this aim are as follows. Section 2 introduce three new bivariate beta exponential (BBE) models based on different copula functions. Parametric and semiparametric methods are used to estimate the parameters of BBE models in Section 3. In Section 4, goodness of fit test for the three models of bivariate beta exponential (BBE) models computed to check the flexibility of different models based on different copula functions. Finally, Simulation is studied to illustrate the performance of the suggested bivariate models and compare each one to other bivariate models in Section 5.

II. Bivariate BE Distributions Based On Copulas

For the bivariate case, copulas are used to link two marginal distributions with joint distribution such that for every bivariate distribution function $F(x_1, x_2)$with continuous marginal $F(x_1), F(x_2)$, there exist a unique copula function C as follows

$$F(x_1, x_2) = C\{F(x_1), F(x_2)\}, \quad (x_1, x_2) \in (-\infty, \infty)X(-\infty, \infty) \tag{11}$$

The density function of bivariate distribution gives as

$$f\ \ (x_1, x_2) = f_1(x_1) f_2(x_2) c(F_1(x_1), F_2(x_2), \tag{12}$$

Where $c(F_1(x_1), F_2(x_2)$ is the density function of copula.

see (Nelsen, 1999). Several copula functions can be used to construct BBE distributions with BE marginals given by (4). In this article, we will applied the Gaussian, Farlie-Gumbel-Morgensten and Plackett copulas to construct BBE distributions.

The joint PDF of X_1 and X_2based on Gaussian copula becomes

$$f\ \ (x_1, x_2) = f_1(x_1) f_2(x_2) \left\{ \frac{1}{\sqrt{1-\rho^2}} (\exp[\frac{-\rho}{2(1-\rho^2)} \{\rho(z_1^2 + z_2^2) - 2z_1\ z_2\ \}]) \right\}. \tag{13}$$

where $\rho \in [-1, 1]$ is a dependence parameter.

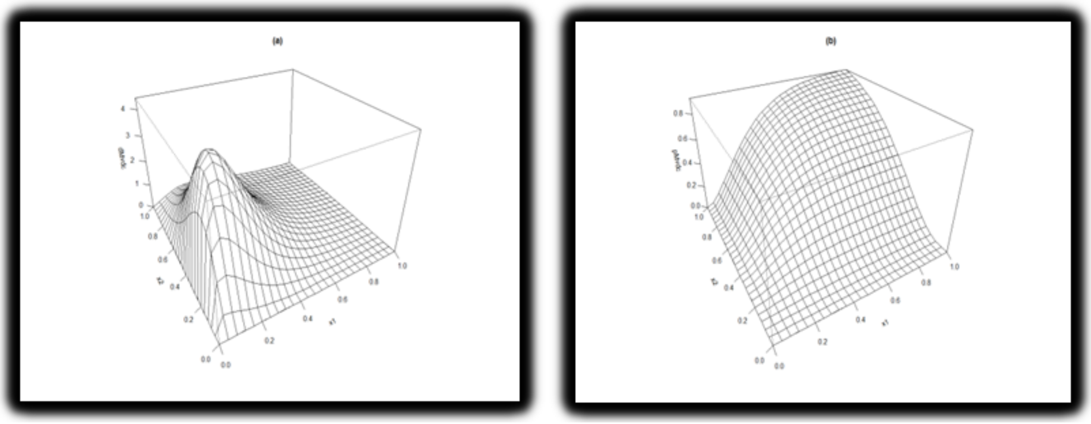

Figure (1): Plots the PDF and Cdf of the BBE based on Gaussian copula

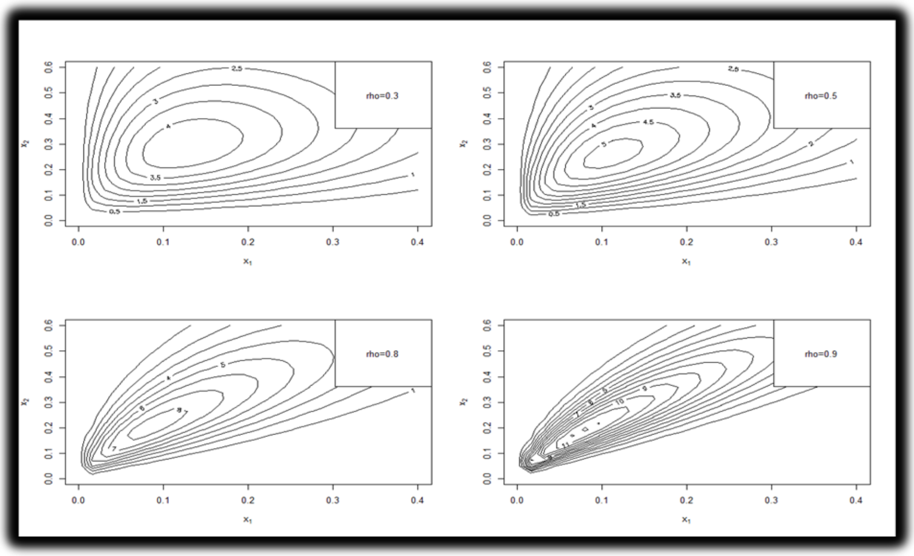

Figure (2): Contour plots of BBEbased on Gaussian copula for different values of θ.

The joint PDF of X_1 and X_2based on Farlie-Gumbel-Morgensten copula becomes

$$f\ (x_1, x_2) = f_1(x_1)f_2(x_2)[1 + \theta(1 - 2u)(1 - 2v)]. \qquad (14)$$

where u and v $\in$ I, and $\theta \in [-1,1]$ is a dependence parameter.

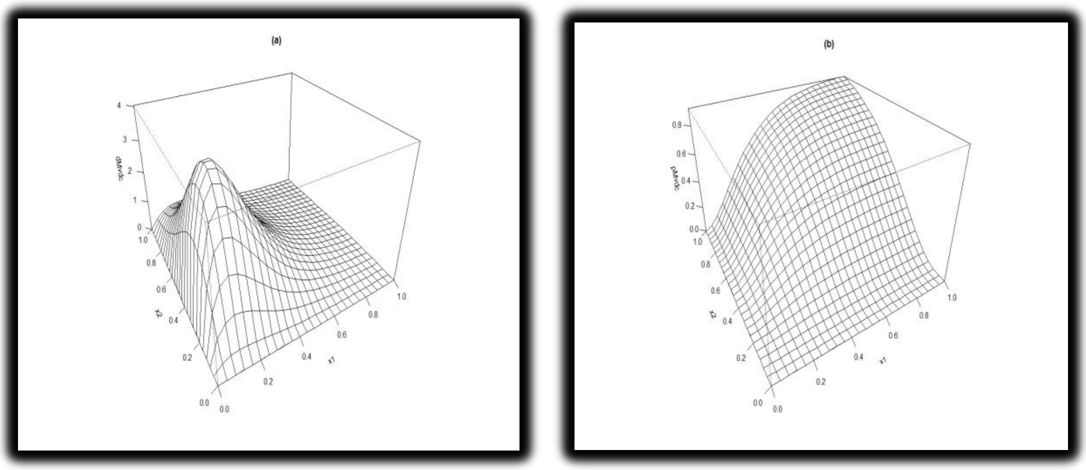

Figure (3): Plots the PDF and Cdf of the BBE based on FGM copula

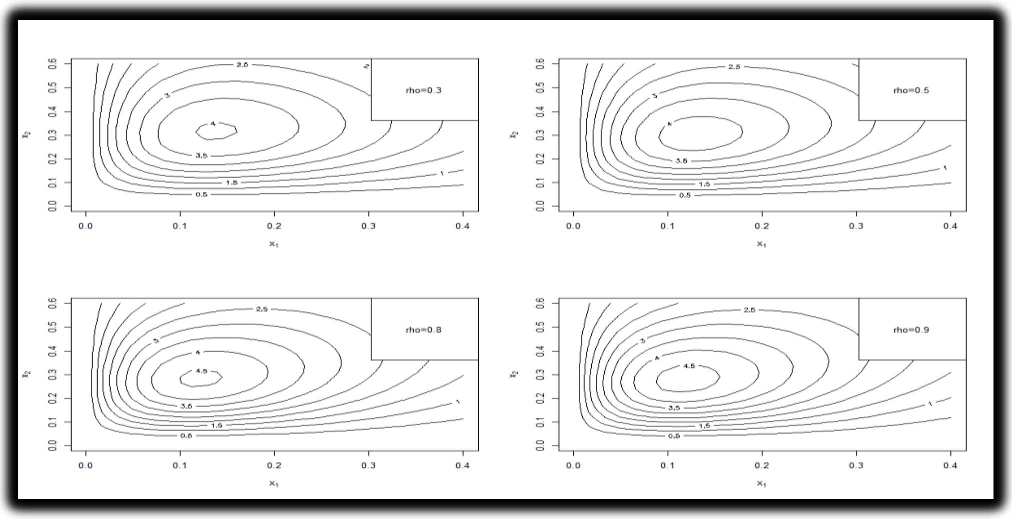

Figure (4): Contour plots of BBEbased on FGM copula for different values of θ.

The joint PDF of X_1 and X_2 based on Plackett copula becomes

$$f\ (x_1, x_2) = f_1(x_1) f_2(x_2) \frac{\theta[1 + (u - 2uv + v)(\theta - 1)]}{\left([1 + (\theta - 1)(u + v)]^2 - 4uv\theta(\theta - 1)\right)^{\frac{3}{2}}}. \tag{15}$$

where u and v $\in$ I, and $\theta \in [0, \infty]$ is a dependence parameter.

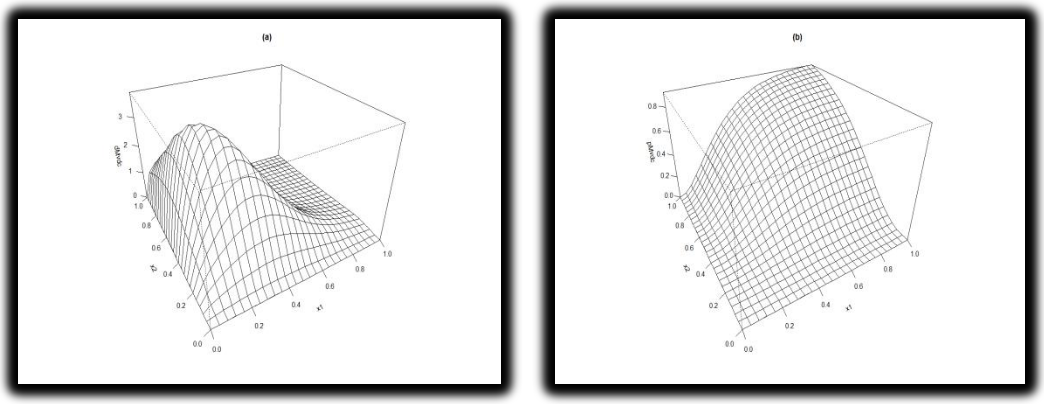

Figure (5): Plots the PDF and Cdf of the BBE based on Plackett copula

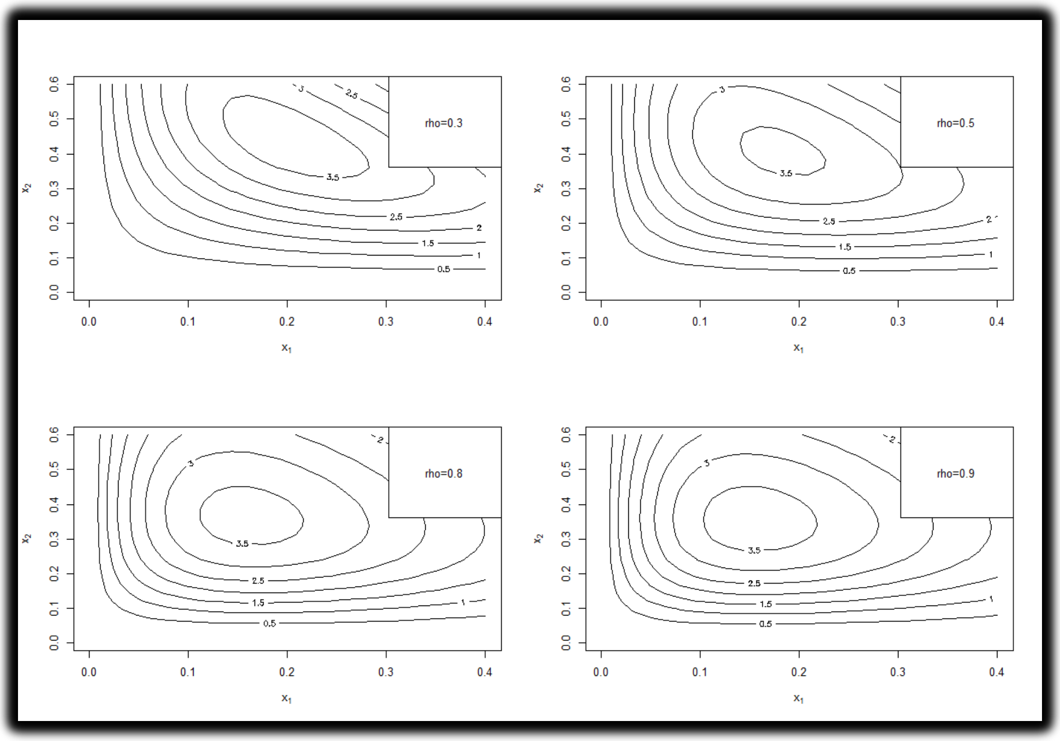

Figure (6): Contour plots of BBEbased on Plackett copula for different values of θ.

III. Parameters Estimation

In this section, we provide the estimation of the unknown parameters of BBE distributions by two approaches to fitting copula models. Parametric and semiparametric are methods used to estimate proposed distribution parameters.

3.1 Parametric methods of estimation:

There are two approaches to fitting copula models. The first one approach is two steps procedure estimating the marginal and the copula parameter separately. The second approach is two steps procedure estimating the marginal and the copula parameter, which is computed from the pseudo-observations separately.

Maximum likelihood estimation (ML)

We provide the estimation of the unknown parameters of BBE distributions by the approach maximum likelihood. By using the two-step estimation (ML). The approach is two steps procedure estimating the marginal and the copula parameter separately.

The log-likelihood function expressed as

$$log\, L = \sum_{i=1}^{n}[log\, f_1(x_{1i}) + log\, f_2(x_{2i}) + log\, c(F_1(x_{1i}), F_2(x_{2i}))] \quad (16)$$

The log-likelihood function in (16) can be re-expressed as

$$log\, L = \sum_{i=1}^{n} log\, f_1(x_{1i}) + \sum_{i=1}^{n} log\, f_2(x_{2i}) + \sum_{i=1}^{n} log\, c(F_1(x_{1i}), F_2(x_{2i}))] \quad (17)$$

The first step is estimating the parameters of marginal distribution F_1 and F_2 by MLE separately as given,

$$\log L_j = \sum_{i=1}^{n} \log f_j(x_{ji})\, , j = 1,2. \quad (18)$$

Then, estimating copula parameters by maximizing the copula density as given

$$\log L = \sum_{i=1}^{n} log\, c(F_1(x_{1i}), F_2(x_{2i})). \quad (19)$$

By considering the first step with (BE) distribution, the parameters of each marginal distribution will be estimated by MLE. If $x_1, \ldots, x_n$ is a random sample fromBE(α_j,a_j,b_j), then the log-likelihood function $L(\alpha_j$,a_j,$b_j)$ is

$logL(x_j,\alpha_j,a_j,b_j) =$

$log(\alpha_j) - log\left(B(a_j, b_j)\right) - b_j\alpha_j x_j +$

$(a_j -$

1log$1-$ $exp-\alpha jxj$

(20)

$$\frac{\partial logL(t_j,\alpha_j,a_j,b_j)}{\partial \alpha_j} = \frac{1}{\alpha_j} - \frac{b_j}{\alpha_j}(\alpha_j x_j) + \frac{(a_j-1)(\alpha_j x_j)\, exp\{-\alpha_j x_j\}}{\alpha_j\, [1-exp\{-\alpha_j x_j\}]}, \quad (21)$$

$$\frac{\partial logL(t_j,\alpha_j,a_j,b_j)}{\partial a_j} = -\{\delta(a_j) - \delta(a_j + b_j)\} + \log\, [1 - exp\{-\alpha_j x_j\}], \quad (22)$$

and

$$\frac{\partial logL(t_j,\alpha_j,a_j,b_j)}{\partial b_j} = -\{\delta(b_j) - \delta(a_j + b_j)\} - \alpha_j x_j \quad (23)$$

The solution of the system of nonlinear equations (21),(22)and (23) gives the MLEs of α_j,$a_j$and b_j.

The maximum likelihood estimates (MLEs)can be calculated by making equations (21),(22)and (23) equal to zero. These equations can be solved numerically for α_j, $a_j$and b_j.Wecanuseiterativetechniquessuch asaNewton-Raphsontypealgorithmtoobtaintheestimatesoftheseparameters.

Then copula density will estimated as given,

$$\log L(\theta) = \sum_{i=1}^{n} log\, c(\widehat{F_1}(x_{1i}), \hat{F}_2(x_{2i})), \quad (24)$$

Where $\widehat{F_1}(x_1)$ and , $\hat{F}_2(x_2)$ denote the ML estimates of the parameters from first step.

The solution of the nonlinear equation (24) gives the MLE of θ.

Modified maximum likelihood estimation (MML)

This a new method is suggested in this article the first step is estimating the parameters of marginal distribution F_1 and $F_2$by MLE separately as given,

$$\log L_j = \sum_{i=1}^{n} \log f_j(x_{ji})\, , j = 1,2.$$

The maximum likelihood estimates (MLEs) can be calculated by making equations (21),(22) and (23) equal to zero. These equations can be solved numerically for α_j, $a_j$and b_j. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the estimates of these parameters.

Second step is estimating copula parameters by maximizing the copula density as given

$$\log L\,(\theta) = \sum_{i=1}^{n} \log\, [c_\theta(\widehat{U_i}, \widehat{V_i})] (25)$$

Where$\widehat{U_i}$, $\hat{V}_i$ are pseudo-observations computed from$\hat{U}_i = \frac{R_{1i}}{n+1} = \frac{n}{n+1}\hat{F}_1\,(t_{1i})$, $\hat{V}_i = \frac{R_{2i}}{n+1} = \frac{n}{n+1}\hat{F}_1\,(t_{2i})$, , $R1i$, $R2i$ are respectively the ranks of $t1i$, $t2i$.

It is important to respect that the margins Cdf.s are estimated parametrically from the first step.

3.2 Semiparametric methods of estimation

Two semiparametric methods to estimate the copula parameter in copula models are compared the two Methods-of-moments approaches of namely inversion Kendall's and inversion of Spearman's rho.

Methods-of-moments

Method-of-moments approaches of inversion Kendall's and inversion of Spearman's rho

As it is mentioned in Kojadinovic and Yan (2010) , let c be a bivariate random sample from CdfC_θ $[F_1(t_1), F_2(t_2)]$, where F_1 and F_2 are continuous Cdf.s and C_θ is an absolutely continuous copula such that $\theta \in \mathcal{O}$, where $\mathcal{O}$ is an open subset of R^2. Furthermore, let $R_1, \ldots, R_n$are the vectors of ranks associated with $t_1, \ldots, t_n$ unless otherwise stated. In what follows, all vectors are row vectors. Method-of-moments approaches are based on the inversion of a consistent estimator of a moment of the copula C_θ. The two best-known moments, Spearman's rho and Kendall's tau, are respectively given by

$$\rho(\theta) = 12\int_{[0,1]^2} \mathrm{u\,v}dC_\theta(u, v) - 3, \quad (26)$$

and

$$\tau(\theta) = 4\int_{[0,1]^2} C_\theta(u, v)dC_\theta(u, v) - 1. \quad (27)$$

Consistent estimators of these two moments can be expressed as

$$\rho_n = \frac{12}{n(n+1)(n-1)}\sum_{i=1}^{n} R_{i,1}R_{i,2} - 3\frac{n+1}{n-1}, \quad (28)$$

And

$$\tau_n = \frac{4}{n(n-1)}\sum_{i=1}^{n} 1[t_{i,1} \leq t_{j,1}]1[t_{i,2} \leq t_{j,2}] - 1. \quad (29)$$

When the functions ρ and τ are one-to-one, consistent estimators of θ are given by

$$\theta_{n,\rho} = \rho^{-1}(\rho_n),$$

$$\theta_{n,\tau} = \tau^{-1}(\tau_n).$$

It can be called inversion of Kendall's (itau) and inversion of Spearman's rho (irho) respectively. For more information, see Kojadinovic and Yan (2010).

As explained above the methods-of-moments (itau) and (irho) estimation method for copula is considered as a semiparametric method of estimation.

IV. Goodness Of Fit Tests For Copula

The idea of this test is to compare the empirical copula with the parametric estimator derived under the null hypothesis see Dobrić and Schmid(2007) and Fermanian(2005). That is, test if C is well-represented by a specific copula C_θ

$$H_0: C = C_\theta \quad Vs. \quad H_1: C \neq C_\theta$$

Two approaches are commonly used in the literature to test the goodness of fit of a copula; the parametric bootstrap see Genest and Rémillard (2008)or the fast multiplier approach see Genest, et al. (2009), and Kojadinovicet al. (2011). The goodness of fit tests based on the empirical process

$$\mathbb{C}_n(u, v) = \sqrt{n}\{C_n(u, v) - C_{\theta_n}(u, v)\},$$

where $C_n(u, v)$ is the empirical copula of the data of T_1 and T_2

$$C_n(u, v) = \frac{1}{n}\sum_{i=1}^{n} 1(U_{i,n} \leq u, V_{i,n} \leq v), \; u, v \in [0,1],$$

$U_{i,n}, V_{i,n}$are pseudo observations from C calculated from data as follows

$U_{i,n} = \frac{R_{1i}}{n+1}$, $V_{i,n} = \frac{R_{2i}}{n+1}$, R_{1i} , R_{2i} are respectively the ranks of t_{1i}, t_{2i}.

Here $C_n(u, v)$ is a consistent estimator and θ_n is an estimator of θ obtained using the pseudo observations. According to Genest et al.(2009), the test statistics is the Cramer-von Miss and is defined as

$$S_n = \sum_{i=1}^{n}\{C_n(U_{i,n}, V_{i,n}) - C_{\theta_n}(U_{i,n}, V_{i,n})\}^2$$

See for details Genest et al., (2009), Genest and Rémillard, (2008) and Kojadinovic et al., (2011).

V. Simulation Data

In this section, a comparison between the three proposed models via different types of copulas is presented. The correlation measures Kendall's tau and Spearman's rho of two variables with BBE distribution are obtained and used to provide the values of copula parameters.

Considering the following values of marginal and copula parameters of BBE distribution based on Gaussian, FGM and Plackett copulas with different sizes of sample (n = 20, 30, 50, 100,150 and 200):

$a_1 = 2, b_1 = 12, \alpha_1 = 2, a_2 = 3, b_2 = 6, \alpha_2 = 2$Gaussiancopula parameter$\theta_G = 0.8$, FGM copula parameter$\theta_F = 0.3$and Plackett copula parameter$\theta_p = 0.3$.

The estimates for these parameters of three models by different three types of copulas and the corresponding bias, mean squared errors and relative mean squared errors based on 1000 replications are reported in Table 1, 2, 3,4,5, and 6.

Table 1. The estimates, the bias, the mean squared errors and the relative mean squared errors of parameters by simulation study for BBE distribution based Gaussian copula

Sample Size		Estimates, bias, mean square errors and relative mean square errors of Parameters						
		$a_1 = 2$	$b_1 = 12$	$\alpha_1 = 2$	$a_2 = 3$	$b_2 = 6$	$\alpha_2 = 2$	$\theta_G = 0.8$
n=20	ML	2.300 0.301 0.688 0.334	11.902 0.098 0.835 0.070	0.049 0.009 0.003 0.065	1.258 0.158 0.383 0.383	5.858 0.142 0.269 0.045	0.013 0.001 0.001 0.119	0.779 0.021 0.000 0.001
	MML	2.300 0.301 0.688 0.334	11.902 0.098 0.835 0.070	0.049 0.009 0.003 0.065	1.258 0.158 0.383 0.383	5.858 0.142 0.269 0.045	0.013 0.001 0.001 0.119	0.861 0.061 0.004 0.005
n=30	ML	2.214 0.214 0.423 0.778	11.833 0.167 1.494 0.458	0.051 0.011 0.005 0.486	1.099 0.099 0.173 0.636	5.912 0.088 0.083 0.187	0.011 0.000 0.000 0.441	0.709 0.091 0.008 0.140
	MML	2.214 0.214 0.423 0.778	11.833 0.167 1.494 0.458	0.051 0.011 0.0053 0.486	1.099 0.099 0.173 0.636	5.912 0.088 0.083 0.187	0.011 0.000 0.000 0.441	0.792 0.007 0.000 0.052
n=50	ML	2.123 0.451 0.970 0.132	11.689 1.142 2.543 0.212	0.048 0.400 0.213 0.107	1.048 0.655 0.565 0.042	5.762 0.877 0.487 0.022	0.010 0.056 0.056 0.009	0.692 0.398 0.159 0.015
	MML	2.123 0.451 0.970 0.132	11.689 1.142 2.543 0.212	0.048 0.400 0.213 0.107	1.048 0.655 0.565 0.042	5.762 0.877 0.487 0.022	0.010 0.056 0.056 0.009	0.713 1.174 0.375 0.009
n=100	ML	2.061 0.832 1.139 0.042	11.759 0.885 1.492 0.124	0.044 0.195 0.070 0.035	1.026 0.348 0.240 0.018	5.903 1.319 0.645 0.008	0.009 0.005 0.005 0.000	0.763 0.505 0.069 0.002
	MML	2.061 0.832 1.139 0.042	11.759 0.885 1.492 0.124	0.044 0.195 0.070 0.035	1.026 0.348 0.240 0.018	5.903 1.319 0.645 0.008	0.009 0.005 0.005 0.000	0.776 0.320 0.102 0.000
n=150	ML	2.035 0.477 0.720 0.360	11.748 0.927 2.083 0.638	0.044 0.203 0.077 0.523	1.014 0.187 0.150 0.150	5.721 1.027 0.649 0.398	0.010 0.003 0.003 0.025	0.790 0.141 0.020 0.025
	MML	2.035 0.477 0.720 0.360	11.748 0.927 2.083 0.638	0.044 0.203 0.077 0.523	1.014 0.187 0.150 0.150	5.721 1.027 0.649 0.398	0.010 0.003 0.003 0.025	0.792 0.412 0.046 0.058
n=200	ML	2.028 0.383 0.563 0.021	11.765 0.865 1.440 0.120	0.043 0.131 0.058 0.029	1.014 0.187 0.412 0.008	5.636 1.338 0.888 0.040	0.010 0.003 0.003 0.000	0.780 0.272 0.074 0.001
	MML	2.028 0.383 0.563 0.021	11.765 0.865 1.440 0.120	0.043 0.131 0.058 0.029	1.014 0.187 0.412 0.008	5.636 1.338 0.888 0.040	0.010 0.003 0.003 0.000	0.789 0.136 0.019 0.000

Table 2. The estimates, the bias, the mean squared errors and the relative mean squared errors of parameters of correlation parameter by simulation study for BBE distribution based Gaussian copula

Sample Size	$\theta_G = 0.8$				
	Estimates	bias	MSE	RMSE	Method Estimation
n=20	0.779 0.861 0.855 0.856	0.021 0.061 0.055 0.056	0.000 0.004 0.003 0.003	0.001 0.005 0.004 0.004	ML MML Itau IRho
n=30	0.709 0.792 0.778 0.780	0.091 0.007 0.022 0.019	0.008 0.000 0.000 0.000	0.140 0.052 0.108 0.088	ML MML Itau IRho

n=50	0.692	0.398	0.159	0.015	ML
	0.713	1.174	0.375	0.009	MML
	0.680	0.442	0.195	0.018	Itau
	0.668	0.487	0.327	0.022	IRho
n=100	0.763	0.505	0.069	0.002	ML
	0.776	0.320	0.102	0.000	MML
	0.749	0.695	0.131	0.003	Itau
	0.742	0.788	0.169	0.004	IRho
n=150	0.790	0.141	0.020	0.025	ML
	0.792	0.412	0.046	0.058	MML
	0.782	0.249	0.062	0.062	Itau
	0.780	0.273	0.074	0.093	IRho
n=200	0.780	0.272	0.074	0.001	ML
	0.789	0.136	0.019	0.000	MML
	0.786	0.183	0.034	0.000	Itau
	0.783	0.224	0.050	0.000	IRho

Table 3. The estimates, the bias, the mean squared errors and the relative mean squared errors of parameters by simulation study for BBE distribution based FGM copula

Sample Size		Estimates ,bias, mean square errors and relative mean square errors of Parameters						
		$a_1 = 2$	$b_1 = 12$	$\alpha_1 = 2$	$a_2 = 3$	$b_2 = 6$	$\alpha_2 = 2$	$\theta_F = 0.3$
n=20	ML	2.364	11.924	0.050	1.161	5.868	0.012	0.503
		0.364	0.076	0.010	0.161	0.132	0.000	0.203
		1.245	0.798	0.005	0.258	0.188	0.000	0.041
		0.623	0.066	0.113	0.258	0.0312	0.067	0.137
	MML	2.364	11.924	0.050	1.161	5.868	0.012	0.523
		0.364	0.076	0.010	0.161	0.132	0.000	0.223
		1.245	0.798	0.005	0.258	0.188	0.000	0.050
		0.623	0.066	0.113	0.258	0.0312	0.067	0.166
n=30	ML	2.207	11.804	0.050	1.115	5.917	0.011	0.034
		0.207	0.195	0.010	0.115	0.083	0.001	0.004
		0.526	1.823	0.005	0.240	0.149	0.001	0.000
		0.263	0.152	0.120	0.240	0.025	0.056	0.001
	MML	2.207	11.804	0.050	1.115	5.917	0.011	0.188
		0.207	0.195	0.010	0.115	0.083	0.001	0.158
		0.526	1.823	0.005	0.240	0.149	0.001	0.025
		0.263	0.152	0.120	0.240	0.025	0.056	0.834
n=50	ML	2.113	11.734	0.048	1.066	5.771	0.010	0.328
		0.113	0.266	0.008	0.066	0.229	0.000	0.028
		0.203	2.145	0.005	0.049	0.123	0.000	0.001
		0.101	0.179	0.120	0.049	0.021	0.013	0.003
	MML	2.113	11.734	0.048	1.066	5.771	0.010	0.366
		0.113	0.266	0.008	0.066	0.229	0.000	0.066
		0.203	2.145	0.005	0.049	0.123	0.000	0.004
		0.101	0.179	0.120	0.049	0.021	0.013	0.014
n=100	ML	2.050	11.804	0.043	1.031	5.890	0.010	0.265
		0.679	0.720	0.140	0.416	0.405	0.005	0.480
		1.086	1.049	0.060	0.253	0.796	0.005	0.063
		0.040	0.087	0.030	0.019	0.010	0.000	0.004
	MML	2.050	11.804	0.043	1.031	5.890	0.010	0.331
		0.679	0.720	0.140	0.416	0.405	0.005	0.423
		1.086	1.049	0.060	0.253	0.796	0.005	0.179
		0.040	0.087	0.030	0.019	0.010	0.000	0.003
n=150	ML	2.036	11.852	0.042	1.022	5.750	0.010	0.083
		0.486	0.543	0.121	0.309	0.921	0.003	0.799
		0.702	1.163	0.058	0.167	0.549	0.003	0.639
		0.026	0.097	0.029	0.012	0.025	0.000	0.157
	MML	2.036	11.852	0.042	1.022	5.750	0.010	0.130
		0.486	0.543	0.121	0.309	0.921	0.003	0.625
		0.702	1.163	0.058	0.167	0.549	0.003	0.391
		0.026	0.097	0.029	0.012	0.025	0.000	0.096
n=200	ML	2.026	11.827	0.042	1.020	5.672	0.010	0.193
		0.351	0.638	0.081	0.269	1.207	0.003	0.395
		0.520	2.891	0.050	0.434	0.726	0.003	0.156
		0.019	0.065	0.025	0.009	0.033	0.000	0.038
	MML	2.026	11.827	0.042	1.020	5.672	0.010	0.234
		0.351	0.638	0.081	0.269	1.207	0.003	0.89
		0.520	2.891	0.050	0.434	0.726	0.003	0.219
		0.019	0.065	0.025	0.009	0.033	0.000	0.015

Table 4. The estimates, the bias, the mean squared errors and the relative mean squared errors of parameters of correlation parameter by simulation study for BBE distribution based FGM copula

Sample Size	$\theta_F = 0.3$				
	Estimates	bias	MSE	RMSE	Method Estimation
n=20	0.503	0.203	0.041	0.137	ML
	0.523	0.223	0.050	0.166	MML
	0.331	0.032	0.001	0.003	Itau
	0.402	0.101	0.010	0.034	IRho
n=30	0.034	0.004	0.000	0.001	ML
	0.188	0.158	0.025	0.834	MML
	0.155	0.125	0.016	0.522	Itau
	0.139	0.109	0.012	0.400	IRho
n=50	0.328	0.028	0.001	0.003	ML
	0.366	0.066	0.004	0.014	MML
	0.246	0.054	0.003	0.010	Itau
	0.310	0.010	0.000	0.000	IRho
n=100	0.265	0.480	0.063	0.004	ML
	0.331	0.423	0.179	0.003	MML
	0.290	0.453	0.056	0.000	Itau
	0.319	0.258	0.067	0.001	IRho
n=150	0.083	0.799	0.639	0.157	ML
	0.130	0.625	0.391	0.096	MML
	0.105	0.717	0.514	0.127	Itau
	0.124	0.648	0.420	0.103	IRho
n=200	0.193	0.395	0.156	0.038	ML
	0.234	0.89	0.219	0.015	MML
	0.213	1.177	0.377	0.025	Itau
	0.225	1.013	0.279	0.019	IRho

Table 5. The estimates, the bias, the mean squared errors and the relative mean squared errors of parameters by simulation study for BBE distribution based Plackett copula

Sample Size		Estimates ,bias, mean square errors and relative mean square errors of Parameter						
		$a_1 = 2$	$b_1 = 2$	$\alpha_1 = 2$	$a_2 = 3$	$b_2 = 2$	$\alpha_2 = 2$	$\theta_p = 0.3$
n=20	ML	2.373	11.897	0.053	1.173	5.859	0.012	0.129
		0.373	0.103	0.013	0.173	0.141	0.000	0.171
		1.203	1.025	0.006	0.413	0.214	0.001	0.029
		0.602	0.085	0.162	0.413	0.036	0.070	0.098
	MML	2.373	11.897	0.053	1.173	5.859	0.012	0.175
		0.373	0.103	0.013	0.173	0.141	0.000	0.125
		1.203	1.025	0.006	0.413	0.214	0.001	0.016
		0.602	0.085	0.162	0.413	0.036	0.070	0.052
n=30	ML	2.219	11.793	0.050	1.095	5.916	0.010	0.449
		0.807	0.759	0.495	1.291	1.136	0.023	0.549
		1.521	1.576	0.198	1.145	0.801	1.258	0.301
		0.207	0.131	0.100	0.085	0.010	0.001	0.074
	MML	2.219	11.793	0.050	1.095	5.916	0.010	0.439
		0.807	0.759	0.495	1.291	1.136	0.023	0.511
		1.521	1.576	0.198	1.145	0.801	1.258	0.261
		0.207	0.131	0.100	0.085	0.010	0.001	0.064
n=50	ML	2.129	11.736	0.049	1.060	5.782	0.010	0.468
		0.474	0.973	0.428	0.817	0.802	0.013	0.617
		0.830	0.250	2.130	0.250	1.187	0.013	0.380
		0.113	0.178	0.125	0.039	0.015	0.001	0.094
	MML	2.129	11.736	0.049	1.060	5.782	0.010	0.357
		0.474	0.973	0.428	0.817	0.802	0.013	0.775
		0.830	0.250	2.130	0.250	1.187	0.013	0.164
		0.113	0.178	0.125	0.039	0.015	0.001	0.011
n=100	ML	2.063	11.802	0.043	1.024	5.900	0.009	0.303
		0.859	0.729	0.174	0.331	1.350	5.373	0.129
		1.094	1.188	0.066	0.248	0.695	5.373	0.017
		0.547	1.340	0.453	0.248	0.426	0.044	0.015
	MML	2.063	11.802	0.043	1.024	5.900	0.009	0.326
		0.859	0.729	0.174	0.331	1.350	5.373	0.351
		1.094	1.188	0.066	0.248	0.695	5.373	0.123
		0.547	1.340	0.453	0.248	0.426	0.044	0.111
n=150	ML	2.044	11.774	0.045	1.019	5.735	0.010	0.455
		0.596	0.830	0.232	0.260	0.975	2.100	0.571
		0.741	1.906	0.099	0.153	0.586	2.100	0.326

		0.027	0.159	0.050	0.011	0.027	0.000	0.080
	MML	2.044 0.596 0.741 0.027	11.774 0.830 1.906 0.159	0.045 0.232 0.099 0.050	1.019 0.260 0.153 0.011	5.735 0.975 0.586 0.027	0.010 2.100 2.100 0.000	0.406 0.389 0.151 0.037
n=200	ML	2.030 0.400 0.551 0.020	11.704 1.090 1.970 0.164	0.044 0.198 0.074 0.037	1.019 0.262 0.406 0.008	5.647 1.297 0.877 0.040	0.010 3.225 3.295 0.000	0.461 0.593 0.601 0.087
	MML	2.030 0.400 0.551 0.020	11.704 1.090 1.970 0.164	0.044 0.198 0.074 0.037	1.019 0.262 0.406 0.008	5.647 1.297 0.877 0.040	0.010 3.225 3.295 0.000	0.456 0.573 0.329 0.081

Table 6. The estimates, the bias, the mean squared errors and the relative mean squared errors of correlation parameter by simulation study for BBE distribution based Plackett copula

Sample Size	$\theta_p = 0.3$				
	Estimates	bias	MSE	RMSE	Method Estimation
n=20	0.129 0.175 0.129 0.154	0.171 0.125 0.171 0.146	0.029 0.016 0.029 0.021	0.098 0.052 0.098 0.071	ML MML Itau IRho
n=30	0.449 0.439 0.403 0.419	0.549 0.511 0.380 0.438	0.301 0.261 0.145 0.199	0.074 0.064 0.036 0.047	ML MML Itau IRho
n=50	0.468 0.357 0.340 0.353	0.617 0.775 0.546 0.717	0.380 0.164 0.081 0.140	0.094 0.011 0. 005 0.009	ML MML Itau IRho
n=100	0.303 0.326 0.331 0.339	0.129 0.351 0.429 0.529	0.017 0.123 0.178 0.076	0.015 0.111 0.161 0.253	ML MML Itau IRho
n=150	0.455 0.406 0.425 0.434	0.571 0.389 0.462 0.495	0.326 0.151 0.213 0.245	0.037 0.080 0.052 0.060	ML MML Itau IRho
n=200	0.461 0.456 0.460 0.465	0.593 0.573 0.590 0.608	0.601 0.329 0.347 0.369	0.087 0.081 0.086 0.091	ML MML Itau IRho

From the results in Table 1,2, 3, 4, 5, and 6 we observed that

1. As expected, most results improve with increases in sample size.
2. For most selected values of $a_1, b_1,\ \alpha_1, a_2, b_2,\ \alpha_2$ and θ the bias, MSE and RMSE of the estimates $\hat{a}_1, \hat{b}_1, \hat{\alpha}_1, \hat{a}_2, \hat{b}_2, \hat{\alpha}_2$ and $\hat{\theta}$ become smaller as the sample size increases.
3. For a_2 greater than a_1, the most results $\hat{a}_1$for are generally better than$\hat{a}_2$ for Furthermore, the values of $\hat{a}_1$ get better more rapidly than the values of $\hat{a}_2$ as the sample size increases.
4. For b_2 greater than b_1, the most results $\hat{b}_1$for are generally better than$\hat{b}_2$for Furthermore, the values of $\hat{b}_1$ get better more rapidly than the values of $\hat{b}_2$ as the sample size increases.
5. the efficient estimators of marginal parameters of three models differ according to the parameters. It seems that ML estimates$\hat{a}_1, \hat{b}_1, \hat{\alpha}_1, \hat{a}_2, \hat{b}_2, \hat{\alpha}_2$and of three models are the same corresponding MML estimates.
6. For copula parameter, the MML provided efficient most estimates for Gaussian, FGM, and Plackett copula parameters compared to ML, Itau, and Irho. It is also noted that the ML and MML estimates for all copula parameters are close.
7. For copula parameter, it is observed that most estimates of Gaussian copula parameter θ_G were more efficient than the corresponding the estimates of copula parameterFGM θ_F and Plackett copula parameter θ_p.

To check if the selected parmetric copula functions are suitable for the marginals, goodness of fit test statisticsusing selected copula functions for the marginals is preformed.The results in Table (7) show a non signficant p-value obtained using parmetric bootstrap for all copula functions which indicate that selected parmetric copula functions provide approporiate fit to the marginals. In addition, estimate of the copula parmeter based on ML, MML, Itau, and Irho methods for the gussian, FGM, and Plackett copulas . This estimatesare used as intial value when fitting these copula models using BE marginals.

Table 7. Goodness of fit test statistics with their p-values and estimate of the copula parameter for selected copula functions.

Copula Function	statistic	p-value	Estimate of copula parameter θ	Method estimation
Gaussian	0.0139	0.6179	0.7986	Ml
	0.0139	0.6578	0.7986	MML
	0.0154	0.5679	0.7816	Itau
	0.0157	0.5809	0.7799	Irho
FGM	0.0113	0.9915	0.1318	Ml
	0.0113	0.9905	0.1318	MML
	0.0119	0.9885	0.1051	Itau
	0.0115	0.9915	0.1239	Irho
Plackett	0.0279	0.2003	0.4016	Ml
	0.0279	0.1923	0.4016	MML
	0.0241	0.2882	0.4255	Itau
	0.0229	0.3601	0.4344	Irho

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