IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 3 Ver. V (May - June 2017), PP 43-48
www.iosrjournals.org

Linear Feedback Observable Systems and Stabilizability

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Abstract: *In this paper, conditions under which linear control systems which are completely observable are stabilizable are investigated. Such systems are shown to be stabilizable provided they satisfy special computable conditions.*
Keywords: *Controllability, Observability, Stabilizability*

I. Introduction

We shall study in E^n the stabilizability of linear control systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \\ y(t) &= H\,x(t) \end{aligned}\right\} \qquad (1.1)$$

where we take $x \in E^n$ as the state, $u \in E^m$ as the input, $y \in E^p$ as the output, where we also take A, B and H respectively as n x n, n x m and p x n constant matrices . Note that we shall regard n, m and p as positive integers. The importance of the concept of stabilizability of any control systems that is observable cannot be quantified on account of its intrinsic theoretical interest to Mathematicians and applied Economists. Its importance also lies in application to design engineering and business management. The importance of stabilizabilty of systems finds its applications in Political Sciences too, where those in power usually contend with many seeming variables which are not related. Such variables like national resource distribution, State and Local Government administrations, which constitute variable co-ordinates of the country under consideration.

In each of the above situations, however , we have to urgently point out that the stabilizability methods outlined in this paper may not be applicable directly to these rather more complex problems which are non-linear systems. But in each case, however, the subject of stabilizability of solutions may still be considered by approximate methods solutions, either by linearization or by higher functional analytic methods or by more sophisticated algebraic concepts.

Note that stabilizability of systems of the form (1.1) has been studied extensively. We can see for instance [1], [2] and other relevant references. In each of these cases, the proofs of the results are quite involved and big. In this paper, we try to prove a shorter and rather easier proof of the stabilizability conditions for the systems (1.1) and so to make it more accessible to a greater number of people who may require the results. For any group of persons who need the stabilizability of control systems, the knowledge of some aspect of Functional Analysis, and of course, basic linear Algebra is very important. The stabilizability of linear control systems touches virtually all the aspects of systems theory. As pointed by LaSalle [2], an important condition for linear feedback stabilizability is controllability. Obviously, a state space E^n is such that a given control systems can be divided into sub-systems; some of which are controllable and others uncontrollable. As pointed out also by LaSalle [2], the entire systems is still stabilizable provided its uncontrollable sub-systems is asymptotically stable. So, we shall not be concerned with the nature of control systems mentioned, rather we require that the given initial state $x(t_0) = x_0$, say, the systems (1.1) has a solution under suitable admissibility conditions on the part of the input u. Furthermore, it should be expected following the well-known duality principle between controllability and observability (see Lee & Markus [3]) that observability should feature in stabilizability.

This paper is made into three major sections. Apart from introductory remarks which we refer to as section 1, we have preliminaries as Section 2. That section aims at explaining the relevant terms exactly and will also assemble facts in the form of lemmas or theorems that are very crucial to establish the main result that will be treated in section 3.Thus we have:

II. Preliminaries

Let us look into the linear differential systems

$$\dot{x}(t) = Ax(t) \qquad (2.1)$$

in E^n where A is a constant n x n matrix. We want to establish the stabilizability of the control systems (1.1) in relation to the stability conditions for systems of the form (2.1). Let us first of all have some important definitions that will help us in this paper. Note that the stability and the asymptotic stability of solutions of the systems (2.2) are well known. We start with the following definitions and propositions.

Proposition 2.1 [Kalman's rank condition].
The systems (1.1) with state $x \in E^n$ is controllable if and only if

$$rank\ [\ B\ AB\ A^2\ \dots\ A^{n-1}B\ ] = n \tag{2.2}$$

Definition 2.1 (Controllability)

The control systems (1.1) is said to be Euclidean controllable if for each $x_0 \in E^n$ and $x_1 \in E^n$, there exists a finite time $t_1 \geq 0$ and admissible control u such that the solution $x\,(t)$, say, of (1.1) satisfies $x\,(t_0) = x_0$ and $x(t_1) = x_1$.
Note that in the definition 2.1 above, if $x_1 = 0$, we say that the given systems is null-controllable.

Lemma 2.1

If $X(t)$ is a fundamental matrix of the systems (1.1), then for $\eta \in E^n$ some non-zero vector, (1.1) is Euclidean controllable on $(0\,, t_1)$ if and only if $\eta^T X(\,t_1)B = 0$ implies that $\eta = 0,$ where $(.)^T$ stands for matrix transpose.

Now let us consider the solution $x(t)$ of the control systems (1.1) with initial state x_0 . If the control function (input) is known from time $t = t_1$, say, and the initial state x_0 is uniquely determined for each output y, we say that the initial state x_0 is observable. Formally, we have the following definition;

Definition 2.2(Observability)

The systems (1.1) with the state x, input u and output y, is said to be observable if for each initial point $x(0) = x_0\,, say, at\ the\ arbitrary\ time\ t_0$, there exists $t_1 \geq t_0$ such that for each admissible input u and output y with $t_0 \leq t \leq t_1$, the state x_0 can be determined.

Definition 2.3 [Complete Observability].

If all the initial states for the systems (1.1) are observable, then we say that the given system is completely observable.

In short, it is easy to obtain from the above definitions of controllability and observability that the two concepts in a system theory enjoy some mutual duality relationship (see[4].. Because of this, a computational condition derivable from the famous Kalman rank condition is also available for observability, which we express as follows;

Proposition 2.2 [Kalman rank condition].

The systems (1.1) is observable if and only if

$$\text{rank}\ [\ H^T\ \ A^T H^T\ \ .\ .\ .\ (A^T)^{n-1}\ H^T\ \ ] = n \tag{2.3}$$

As we know, stabilizability is derivable from stability. In short, as pointed out by LaSalle [2], linear systems (1.1) can be stabilized if there exists a matrix K such that the systems

$$\dot{x} = (A + BK)x \tag{2.4}$$

is asymptotically stable. This can be stated equivalently as if the matrix $A + BK$ is stable. For this situation, the control $u\,(t)$`is given in the feedback form

$$u\,(t) = Kx(t),\ t \geq 0$$

ensuring the control systems (1.1) assumes the form (2.4). Let us remark as follows;

Remark 2.1

For convenience, we take $Z \in E^{n\,x\,n}$ (a suitable n x n constant matrix) and define $Z = A + BK$. That is Z now becomes any convenient n x n constant matrix. In view of this remark, (2.4) can assume the form

$$\left.\begin{aligned} \dot{x}\,(t) &= Z\,x(t) \\ x(t_0) &= x_0 \\ y(t) &= H\,x(t) \end{aligned}\right\}` \tag{2.5}$$

As a matter of fact, LaSalle [2][5] defines stability of control systems (1.1) in terms of the stability of the differential systems (2.1), in terms of definition which we hereby adopt.

Definition 2.4 (Stabilizability LaSalle [2])

The systems (1.1) or the pair (A , B) is stabilizable if there exists a matrix C such that $(A + BC)$ is stable.

Note: A simple test for asymptotic stability of linear systems of the form (2.5) is that all the eigenvalues of Z have negative real parts [4]

Now, a nice method of determining stabilizability is the use of Lyapunov functions. Any systems for which a suitable Lyapunov function can be constructed is asymptotically stable. For linear systems, however, the process of determining Lyapunov functions is not very difficult. Lyapunov functions are usually defined along the solution paths x, say, of the systems under consideration.

Definition 2.5

A real-valued positive function $V(.)$ whose first order derivative for all non-zero values of x exists is said to be Lyapunov function for the systems (2.5) of which x is a solution, (see [5]).

Now, from (2.5) and by the well known variation of parameter formula, it can be seen that the input y is given by

$$y(t) = H\, e^{Zt}\, x_0 \; + \; H \int_0^t e^{Z(t-s)}\; Bu(s)ds \tag{2.6}$$

Now, let us define

$$\eta(t) = y - H \int_0^t e^{Z(t-s)}\; Bu(s)ds \; = \; He^{Zt}\, x_0 \tag{2.7}$$

So we have

$$\eta(t) = He^{Zt}\; x_0 \tag{2.8}$$

If we now consider the performance of the systems (2.5) over some time interval $[0\,,\, t_1\,]$, we have the observability grammain.

Definition 2.6 [Observability Grammain].

The n x n matrix denoted $M(0\,, t_1)$ and defined by

$$M(0\,, t_1) \; = \; \int_0^{t_1} e^{Z^T s}\; H^T\, H\, e^{Zs}\, ds \tag{2.9}$$

Is called the observability grammain of the systems (2.5) over the interval $[0\,, t_1\,]$.

Remark 2.2

We hereby remark as follows;

(a) The controllability grammain is defined just as in (2.9) with H replaced by B.

(b) If the systems (2.5) is completely observable and we consider the entire interval $[0\,, \infty\,)$, we write the grammain (2.9) as

$$M(0\,, \infty\,) = \int_0^{\infty} e^{Z^T s}\; H^T H\, e^{Zs}\, ds$$

(2.10)

We note that $M\,(0\,, t_1\,)$ is a positive semi-definite n x n matrix. LaSalle [2]. We hereby state the following well known result which is very very important in greater detail for completeness.

Lemma 2.2

A necessary and sufficient condition for the control systems (2.5) to be completely observable on $(\,0\,, t_1\,)$ is that the observability grammain $M(\,0\,, t_1\,)$ is nonsingular (see [6]).

Proof:

(Necessity): We assume the system (2.5) is completely observable; we prove that $M\,(\,0\,, t_1\,)$ is non-singular. But suppose, on the contrary, that $M\,(\,0\,, t_1\,)$ is singular. This means that for some non-zero p-row vector v, say, we have $M(0\,, t_1)v = 0$ then

$$v^T\, M(b\,0\,, t_1)v \; = \; 0$$

(2.11)

Thus, from (2.9), we get (2.11) above as

$$v^T \left\{ \int_0^t E^{Z^T s}\ H^T H\ e^{Zs}\ ds \right\} v\ =\ 0$$

$$\Rightarrow\ \{\int_0^{t_1} v^T\ [(He^{Zs})^T\ (He^{Zs})] v ds\} =\ 0$$

$$\Rightarrow\ He^{Zs}\ v\ =\ 0,\ t \geq 0 \qquad (2.12)$$

Now, from some initial state x_0 in (2.9), taking $v = x_0$, (2.12) together with (2.8) imply that for some non-zero initial state x_0 , the signal η and the output y, each always zero for all non-zero initial point. This therefore contradicts the observability of this systems. Consequently we conclude that $M\ (0\ , t_1\ )$ is non-singular.

(**Sufficiency):** Let us now assume that the grammain $M\ (0\ ,\ t_1)$ is non-singular. From (2.8), we obtain by multiplying each side by $e^{Z^T t}\ H^T,$ thus

$$e^{Z^T t}\ H^T\ H\ e^{Zt}\ x_0\ =\ e^{Z^T t}\ H^T\ \eta(t).$$

Integrating both sides from 0 to t_1 we get

$$\int_0^{t_1} e^{Z^T t}\ H^T\ H\ e^{Zt} x_0 dt\ =\ \int_0^{t_1} e^{Z^T t}\ Z^T\ \eta(t) dt.$$

This, considering (2.9) gives

$$M\ (0\ , t_1\ ) x_0\ =\ \int_0^{t_1} e^{Z^T}\ \eta\ (t) dt.$$

That is

$$x_0\ =\ [M\ (0\ ,\ t_1\ )]^{-1}\ \int_0^{t_1} e^{Z^T t}\ H^T\ \eta(t) dt \qquad (2.13)$$

(2.13) shows that the initial state x_0 can be uniquely determined from the signal η and the output, provided that the grammain $M(0\ , t_1)$ is invertible. This means that the given systems (2.5) is observable, completing the proof of the lemma.

Now, we are ready to face the main result of this paper. However, in doing so, we must strictly adhere to the following very crucial remarks.

Remark 2.3

(a) In what follows, Z in (2.5) above and A + BC of (3.4) below will play an interchangeable roles each being a constant n x n matrix. The only precaution we must take is to obey the laws of matrix transpose for product s of matrices.

(b) As a result of (a) above, all the conditions to be outlined in the following section as a whole concerning Z are transferable to A + BC. For example, the expression (2.9) for the observability grammain can be written simply as

$$M(0\ ,\ t_1\ )\ \int_0^{t_1} e^{(A+BC)^T s}\ H^T\ H\ e^{(A+BC)s} ds \qquad (2.14)$$

We are now ready to state our main result.

III. Main Result

Theorem 3.1 (Stabilizability Theorem).

In E^n , the n-dimensional Euclidean space, consider the linear control systems

$$\left.\begin{aligned} \dot{x}\ (t) =\ & Ax(t)\ +\ Bu(t) \\ x(0) =\ & x_0 \end{aligned}\right\} \qquad (3.1)$$

$$y(t) =\ H\ x(t) \qquad (3.2)$$

For which C, a constant m x n matrix, has a linear feedback control

$$u(t) = \ Cx(t) \tag{3.3}$$

which transforms the control systems (3.1) to

$$\dot{x}\,(t) \ = \ (\ A + BC\ )x(t) \tag{3.4}$$

In (3.1), $x \ \in \ E^n\ ,\ \ u \ \in \ E^m,\ y \ \in E^p$ are vectors and A, B and H are respectively n x n, n x m and p x n constant matrices and the n x n constant matrix A + BC is asymptotically stable. Assuming that the given systems (3.1) is completely observable, then for D, some positive n x n symmetric matrix, the condition for the stabilizability of the systems (3.1)is that

$$A^T D \ + DA \ + \ C^T\, B^T D \ + DBC \ + \ H^T\, H \ = \ 0 \tag{3.5}$$

where $(.)^T$ denotes the matrix transpose.

Proof:

Now, from hypothesis, the asymptotic stability of (2.5) is

$$lim_{t\to\infty}\,|x_1(t) - x_2\,(t)| \ = \ \ 0$$

for the two solutions x_1 and x_2 of (2.5). By (2.10) we obtain the grammain of (2.5) as

$$M(0\,,\infty) \ = \ \int_0^\infty e^{Z^T t}\, H^T\, H\, e^{Zt} dt \tag{3.6}$$

Differentiating (3.6), we get

$$\int_0^\infty \frac{d}{dt}\big[e^{Z^T t}\, H^T\, He^{Zt}\big]dt = \ \int_0^\infty [Z^T e^{Z^T t}\, H^T\, H\, e^{Zt} \ + Ze^{Z^T t}\, H^T H\, e^{Zt}\,]dt \tag{3.7}$$

Now, evaluating this, we get

$$e^{Z^T t} H^T H\, e^{Zt} \ \ |_0^\infty = \ \ e^{Z^T t} H^T\, He^{Zt} dt \ + \ \ Z\int_0^\infty e^{Z^T t}\, H^T He^{Zt} dt.$$

and by asymptotic stability, we get

$$0 \ - \ \ H^T H \ = \ Z^T\, M(0\,,\infty) + \ Z\, M(0\,,\infty).$$

Or

$$Z^T\, M(0\,,\infty) + \ \ Z\, M(0\,,\infty) + \ H^T\, HJ \ = \ \ 0 \tag{3.8}$$

and this is the required stabilizability condition as it applies to equation of type (2.5). Now, defining $D \ = \ M(0\,,\infty)$ and setting Z = A + BC in (3.8), with regards to the remark 2.3(a), we finally get the stabilizability equation for the systems (3.1) as

$$(A + BC)^T D \ + D(A + BC) + \ H^T H \ = 0$$

which reduces to

$$A^T D \ + \ \ DA\ \ C^T B^T D \ + \ \ DBC \ + \ \ H^T H = \ \ 0$$

$$= \ A^T D \ + \ \ DA \ + \ \ C^T B^T D \ + DBC \ + \ H^T H \ = \ \ 0$$

completing the proof of the theorem.

Example.

The following example illustrates the application of the above theorem.
Consider in E^n , the control systems

$$\left.\begin{array}{l} \ddot{x} \ + \ \ 2\dot{x} \ - \ \ x \ = \ \ u \\ y(t) = (3.2, 2.5\,)x(t) \end{array}\right\} \tag{4.1}$$

with feedback

$$u = C\ x(t) \tag{4.2}$$

where $C = (-6, -2)$ (4.3)

Obviously, the systems (4.1) is of the standard form

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= (3.2, 2.5)x \end{aligned}\right\}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $H = (3.2, 2.5)$

By the lemma 2, the given systems is clearly observable. Now, choosing $D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ a positive symmetric 2 x 2 matrix, we can determine a Lyapunov function for the systems (4.1) in terms of H. Also, with C of (4.3), the systems (4.1) is now of the form

$$\dot{x} = (A + BC)x$$

Where

$$A + BC = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}.$$

The eigenvalues of A + BC have negative real parts, thus the systems (4.1) is stabilizable and clearly we seen that

$$A^T D + DA + C^T B^T D + DBC + = -H^T H$$

Or

$$A^T D + DA + C^T B^T D + DBC + H^T H = 0$$

as required.

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S.E. Aniaku. "Linear Feedback Observable Systems and Stabilizability." IOSR Journal of Mathematics (IOSR-JM) 13.3 (2017): 43-48.