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Null-Controllability and Uniqueness of Optimal Trajectory for Controllable Systems

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***Abstract**: In this paper, it is shown that in a linear control system which can be steered to zero target from different initial points, the ultimate trajectory is unique provided the control function is bang-bang.*
***Keywords:** Bang-bang control Null-controllability, optimal trajectory,*

I. Introoduction

Let E^n denote the n-dimensional Euclidean space. In E^n we consider the linear control system

$\dot{x}(t) = A(t)\, x(t) + B(t)u(t)$

(1.1)

$x(0) = x_0$

where $\varepsilon\, E^n$, A and B are $n\ x\ n$ and $n\ x\ m$ continuous linear matrix functions respectively on $E^+ = [0, \infty) = I$, say, and u is an m-vector valued measurable function with values $u(t)$ lying in a compact, convex set Ω of E^m. Such a u is said to be admissible.

A physical system x is said to be controllable if with the aid of external or in-built mechanism (which we call control or input function denoted by u) the system can be transferred from state x_0 , say, to another state x_1 in a finite time $t > 0$. If the state x_1 to which the system is eventually transferred to is zero (that is the origin in the case of the Euclidean space E^n), we say that the system is null-controllable.

Null-controllability is very very important and indeed needed in human situations. For example, the primary aim of many Government and even non-governmental organizations e.g. NAFDAC, WHO, to mention but a few, try to reduce to zero the number of tuberculous patients, the number of HIV/AIDS patients e.t.c in a finite time $t > 0$ with the help of subsidized effective drugs for the ailment. With the advances in the means of transportation e.g. cars, aeroplanes, ships motorcycles, the number of the people x dying in mishaps has been increasing in recent years. Various mechanisms are being put to such means of transportation in a view to reducing to zero the number of causalities x in a finite time $t > 0$.

Generally, if x is any unwanted object or situation in any community, the desirability and urgency to stamp it out (null-controllability) is very very necessary. For instance, x may denote the number of dangerous criminals in a society , frequent power cut in a certain community or the number of building collapsing in a town. Null- controllability of x in each of these situations is not only necessary but very important. This is why the topic of null-controllability has been very important and has an increasing interest to researchers such as Chukwu [1], Schitendorg [2] and Eke [3] For the system (1.1) above, the subset C^m of E^n is the m-dimensional unit cube, where $C^m = \{u: u(t)\varepsilon\Omega, |u_j| \leq 1,\ j = 1,2,3 \dots m\}$.

Note that the absolutely continuous solution of (1.1) will be denoted by $x(t\,, u)$ and is

$$x(t\,, u) = X(t)x_0 + X(t) \int_0^t X^{-1}(s)B(s)u(s)ds \qquad (1.2)$$

where $X(t)$ is a fundamental matrix solution of the system (1.1) for B = 0 and $X(0) = I$, the identity matrix.

The null-controllability, according to Ek e is archived by imposing the condition of (1.1) the boundary condition

$$Tx = 0 \qquad (1.3)$$

Here, it is expected that T is a bounded linear operator defined on $C[E^+ ,\ E^n]$, the space of all bounded and continuous operators from $E^+\ to\ E^n$.

Definition 1

In the control systems as (1.1) above , if the control function u assumes its maximum value or power, then it is called optimal, and so can be denoted by $u^* = \pm 1$. This control u^* is called a bang-bang control.

Definition 2
The systems (1.1) is said to be Euclidean controllable if for each $x_0 \in E^n$ and $x_1 \in E^n$ each, there exists an admissible control u and finite time $t_1 > 0$ such that the solution $x(t, u) = x(t)$ of (1.1) satisfies $x(0) = x_0$ *and* $x(t_1) = x_1$.

Definition 3 In the definition 2 above, if $x_1 = \bar{0}$, then we say, that the systems (1.1) is Euclidean null-controllable.

We say that an object such as control u or any other concept is optimal if it is adjusted to be the best possible in the concept of prevailing circumstances.

Definition 4 (Trajectory).
The path or locus along which the control function $u(t)$ can steer a point from one point to another in the space E^n is called trajectory. This path or locus is usually denoted by G. When this trajectory is a track of optimality achievement, then it is called optimal trajectory.

We now consider the following Theorem which will help us to establish our goal.

Theorem 1 (Lee and Marcus [4])
Consider the autonomous linear process (1.1) in $\mathcal{R}^n$, with compact restraint set $\Omega\, C\, \mathcal{R}^m$,initial state x_0 and the origin as a the fixed target in $\mathcal{R}^n$. Assume

(a) U = 0 lies in the interior of Ω,
(b) (1.1) is controllable ,
(c) A is stable , that is each eigenvalue λ of A satisfies $Re\ \lambda < 0$. Then there exists a minimal time optimal controller $u^*(t)\, C\, \Omega$ on $0 \le t \le t^*$ steering x_0 to the origin.

Proof

We know that in (1.1) (i). u = 0 lies in the interior of Ω, (ii). (1.1) is controllable and (iii). A is stable ; then the domain of null-controllability is in $\mathcal{R}^n$. Then, there exists a controller u(t) $c\, \Omega$ on $0 \le t \le t_1$ steering x_0 to the origin. Because the co-domain is a compact target set G(t) on $0 \le t \le t_1^*$ and controller u(t) $C\, \Omega$ on $0 \le t \le t_1^*$ steering x_0 to G(t_1), then we have an optimal controller $u^*(t)\, C\, \Omega$ on $0 \le t \le t^*$ steering x_0 to $G(t^*)$.

In the problem under study, we shall assume that n = 2 and m = 1 for ease of understanding of the trajectory through which the control function u transfers the systems (1.1) from a given initial point x_0 , say, to the origin of the x – y Cartesian co-ordinate plane. However, not regarding this assumption, the result of this paper can be generalized to arbitrary integral values of n and m. Having these conditions in mind, we are now ready to state and prove our main result of this paper.

1.Main Theorem

In what follows, we shall assume that the control systems is defined in $E^2$and extreme values of u^* is bang-bang control, steering the given systems (1.1) from any suitable initial point x_0 to zero $\bar{0}$ in finite time $t_1 > 0$. The problem we are interested in solving, is there fore, the following:

Theorem.

Consider the control systems (1.1) in E^2, that is

$$\dot{x}(t) = A\dot{x}(t) + Bu(t) \qquad (2.1)$$
$$x(0) = x_0$$

where $x \in E^2$,A and B are respectively 2 x 2 and 2 x 1matrices. If u^* is an admissible optimal control which is bang-bang and steers the systems (2.1) from any initial point along a suitable locus to the origin in a finite time $t_1 > 0$, the resulting ultimate optimal trajectory r^* of null-controllability is unique.

Proof:

We know that the systems (2.1) is controllable. Since our $u^* = \pm 1$ is bang-bang, then it is normal. Then the convex restraint set $E^2 \subset E^m$ contains u = 0 in its interior, and we have the target G as the origin x = 0. We know that (2.1) satisfies the normality condition since u is bang-bang. Then for each point x_0 in the domain of null- controllability E^2, there exists a unique extremal controller $u^*(t)$ steering x_0 to the origin, and $u^*(t)$ is the optimal controller.

Also , if A is stable, then from $E^2 = E^n$, and so each point $x_0 \in E^n$ can be steered to the origin by just one extremal controller $u^*(t)$, namely the optimal controller.

Example.

Consider the autonomous control process in E^2.

$$\dot{x} = Ax + Bu \qquad (2.2)$$
$$x(0) = x_0$$

In which $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,
Then (2.1) takes the standard form

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \qquad (2.3)$$

$$x(0) = 0$$

Note that at extremal point, we have

$$x_2 + u = 0 \xrightarrow{yields} x_2 = -u$$

$$-x_2 + u = 0 \xrightarrow{yields} x_2 = u$$

with restraint set $\{u \in \Omega : |u| \leq 1\}$ in E^1. We wish to synthesize the minimal line $x_1 = 0$ with additional requirement that the response can thereafter be held on this line. Thus the target set is

$$G = Core\,\{x_1 = 0\}.$$

If the response lies on $x_1 = 0$, then $\dot{x_1} = 0$, $x_2(t) = -u(t)$ and so $|x_2| \leq 1$.

Conversely, each point $x_1 = 0$, $|x_2| \leq 1$ can be cntrolled in a set $|x_2| \leq 1$ by

$$u(t) = -x_{2_0} e^{-2t} \quad \text{for } t \geq 0.$$

Thus G is the set $\{x_1 = 0, |x_2| \leq 1\}$. We note that G is a compact convex set in E^2 and also that each point $[x_{0_1}, x_{0_2}] \in$ □ can be steered to G by non-extremal controller

$$u(t) = -x_{2_0} e^{-2t} \quad \text{for } t \geq 0.$$

Using the coefficient matrices $A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the vector w = 1 along Ω, we see that the normality condition is satisfied. Thus (2.2) is controllable and by theorem 1 we assert that the domain of null-controllability is all E^2 and that each initial point in E^2 can be steered to G by a unique extremal controller $u(t)$. This extremal controller is optimal and the track is known as optimal trajectory.

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