IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 3 Ver. V (May - June 2017), PP 73-77
www.iosrjournals.org

Complement Topologies

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Abstract

Let (X, τ) be a topological space. We consider the collection $\tau^c = \{G^c : G \in \tau\}$ of all the τ-closed subsets of X. If τ^c is a topology on X then we call τ a *complement topology* on X. Necessary and sufficient conditions for a topology τ on X to be a complement topology on X are examined. We proved, among other things, that

1. Any finite topology (topology with a finite cardinality) is a complement topology;
2. A topology τ on X is a complement topology on X if, and only if, it is closed under arbitrary intersections;
3. The family of the complements of the topologies in a chain of complement topologies on any set X is itself also a chain of complement topologies on X.

Keywords: *Complement Topology, Arbitrary Intersections, Chain of Topologies.*

I. Introduction

It is known that a topology τ on a set X is the collection of all the *open* subsets of X. Hence, a topology τ on a set X is a collection of subsets of X which satisfy the *axiom of openness*; the standard four conditions. Openness of a subset is therefore relative to the topology under consideration. Some sets which are considered closed in one topology are open in another topology and vice-versa. A question of interest is *Can all those sets considered closed with respect to a topology on a set X be precisely the ones considered open with respect to another topology on X?* Of course, we are excluding the trivial cases of the discrete and indiscrete topologies on X. This seemingly academic but rather interesting question is the main motivation for this paper.

II. Main Results|De_Nitions, Properties And Im- Plications

Definition 2.1 *Let (X, τ) be a topological space and let τ^c be the collection*

$$\tau^c = \{G^c : G \in \tau\}$$

of complements of τ-open sets. Then we call τ^c the complement of the topology τ on X.

Definition 2.2 *If (X, τ) is a topological space and the complement τ^c of τ is itself also a topology on X, we call τ a complement topology, on X.*

REMARK
Since $\tau = (\tau^c)^c$, it follows that τ is a complement topology on X if and only if τ^c is also a complement topology on X. It turns out that large classes of topologies are complement topologies.

Theorem 2.1 *Every topology on a finite set is a complement topology.*

Proof:

Let τ be a topology on a finite set X and let τ^c be its complement. Then

1. Clearly both $\emptyset$ and X belong to τ^c.
2. Let $\{G_i\}_{i=1}^n \subset \tau^c$. Then $\bigcap_{i=1}^n G_i = (\bigcup_{i=1}^n G_i^c)^c$. But $G_i \in \tau^c \Rightarrow G_i^c \in \tau$. $\Rightarrow \bigcup_{i=1}^n G_i^c \in \tau$. $\Rightarrow \bigcap_{i=1}^n G_i = (\bigcup_{i=1}^n G_i^c)^c \in \tau^c$. $\Rightarrow \tau^c$ is closed under finite intersections.
3. Let $\{G_\alpha\}_{\alpha\in\Delta} \subset \tau^c$. Then $\bigcup_{\alpha\in\Delta} G_\alpha = \left(\bigcap_{\alpha\in\Delta} G_\alpha^c\right)^c$. Now, $G_\alpha \in \tau^c \Rightarrow G_\alpha^c \in \tau$. $\Rightarrow \bigcap_{\alpha\in\Delta} G_\alpha^c \in \tau$, as finite intersections of sets of τ belong to τ. (We observe that the intersection cannot be infinite since X is finite.) Hence, since the complement of every set in τ is collected in τ^c, it follows that $\bigcup_{\alpha\in\Delta} G_\alpha = \left(\bigcap_{\alpha\in\Delta} G_\alpha^c\right)^c \in \tau^c$. This implies that τ^c is also closed under arbitrary unions. Hence the complement of every topology on a finite set is a topology on the set.

Example 2.1

Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be any non-empty finite set and let

$$\tau = \{\emptyset, X, \{x_1\}, \{x_1, x_2\}\}$$

be a topology on X. Then $\tau^c = \{X, \emptyset, \{x_2, x_3, \cdots, x_n\}, \{x_3, x_4, \cdots, x_n\}\}$ is clearly a topology on X.

Example 2.2

Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be a non-empty finite set and let

$$\tau_k = \{\emptyset, X, \{x_1\}, \{x_1, x_2\}, \cdots, \{x_1, x_2, \cdots, x_k\}\}; 1 \le k < n.$$

Then τ_k is a topology on X, for all k. Now

$${\tau_k}^c = \{X, \emptyset, \{x_2, \cdots, x_n\}, \{x_3, \cdots, x_n\}, \cdots, \{x_{k+1}, \cdots, x_n\}\}$$

is also a topology on X, $1 \le k < n$. (This example illustrates the remark after Definition 2.2 above. More of such examples appear at the end of section 3.) The proof of theorem 2.1 above points the way for a more general result.

Theorem 2.2 *Let X be any nonempty set and let τ be a finite topology (topology with finite cardinality) on X. Then τ is a complement topology on X.*

Corollary 2.1 *Let X be an infinite set and let τ be a topology on X. Then the complement τ^c of the topology τ is itself a topology on X if τ contains only a finite number of open sets.*

Example 2.3

Let $a, b \in R$ be any two real numbers. Then $\tau = \{\emptyset, R, \{a\}, \{b\}, \{a, b\}\}$ is a topology on R. Without loss of generality, we can let $a < b$. Then the complement

$\tau^c = \{\emptyset, R, R - \{a\}, R - \{b\}, R - \{a, b\}\}$

$= \{\emptyset, R, (-\infty, a)\bigcup(a, +\infty), (-\infty, b)\bigcup(b, +\infty), (-\infty, a)\bigcup(a, b)\bigcup(b, +\infty)\}$

of τ is easily seen to be a topology on R.

Let $G_0 = N = \{0, 1, 2, \cdots\}$, $G_1 = \{1, 2, 3, \cdots\}$, $G_2 = \{2, 3, 4, \cdots\}$. Then $\tau = \{\emptyset, G_k\}_{k=0}^2$ is easily seen to be a topology on N. The complement of τ, $\tau^c = \{\emptyset, N, \{0\}, \{0, 1\}\}$ is also a topology on N. In general if $G_0 = N$, $G_1 = N - \{0\}$, $G_2 = N - \{0, 1\}$, $G_3 = N - \{0, 1, 2\}, \cdots$, $G_n = N - \{0, 1, 2, \cdots, n-1\}$, then $\tau = \{\emptyset, G_k\}_{k=0}^n$ is a topology on N, and its complement τ^c is also a topology on N.

Now, every topology τ on a finite set is necessarily finite. Hence theorem 2.2 asserts, relative to theorem 2.1, that every finite topology on an infinite set is a complement topology. This raises the following interesting question: *Are the finite topologies the only topologies on infinite sets that are complement topology?* That is, is a complement topology on an infinite set necessarily finite? The next theorem which answers the above question in the negative provides a characterization of complement topologies.

Theorem 2.3 *A topology τ on a set X is a complement topology if, and only if τ is closed under arbitrary intersections.*

Proof:

$\Longrightarrow$ Clearly if τ is a complement topology then it is closed under arbitrary intersections.

$\Longleftarrow$. Let τ be closed under arbitrary intersections and let τ^c be the complement of τ. We show that τ^c is a topology on X. We need only show that τ^c is closed under arbitrary unions, as the other properties of a topology are easily seen to be satisfied by τ^c. So, let $\{A_\alpha : \alpha \in \Delta\} \subset \tau^c$ be any family of sets of τ^c. We consider

$$\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^c = \bigcap_{\alpha\in\Delta} A_\alpha^c$$

Clearly $A_\alpha^c \in \tau$, for all $A_\alpha \in \tau^c$. Since τ is, by hypothesis, closed under arbitrary intersections $\bigcap_{\alpha\in\Delta} A_\alpha^c \in \tau$. Hence the left side of (1) is an element of τ; implying that $\left[\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^c\right]^c = \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) \in \tau^c$.

From theorem 2.3, it follows that every discrete topology is a complement topology; and in particular it follows that discrete topologies of in_nite sets (which necessarily contain in_nitely many open sets) are complement topolo- gies. And there are other complement topologies, with in_nitely many open sets, which are not discrete topologies.

Lemma 2.1 (Comparison) *Let τ_1 and τ_2 be any two complement topologies on a set X such that (say) τ_1 is weaker than τ_2. Then τ_1^c is weaker than τ_2^c.*

III. Application

Definition 3.1 *A family $C = \{\tau_\alpha\}_{\alpha\in\Delta}$ of topologies on a set, X, is called a chain of topologies, on X, if elements of C are pair-wise comparable, in that for any two topologies, τ_α and τ_β, in C, either τ_α is weaker than τ_β or vice versa.*

Definition 3.2 *Any topology which is an element of a chain C of topologies on a set is called a chain element topology.*

Theorem 3.1 *Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be any non-empty finite set. There exists a finite family of topologies on X forming a chain, such that the family of their complement topologies is also a chain.*

Proof:
Let

$$G_1 = X - \{x_1\} = \{x_2, x_3, \cdots, x_n\}.$$

Then $\tau_1 = \{\emptyset, G_0, G_1\}$ is a topology on X.
Let

$G_0 = X;$

$G_1 = X - \{x_1\} = \{x_2, x_3, \cdots, x_n\};$

$G_2 = X - \{x_1, x_2\} = \{x_3, x_4, \cdots, x_n\}.$

Then $\tau_2 = \{\emptyset, G_k\}_{k=0}^{2}$ is a topology on X, stronger than τ_1.

$$\vdots$$

Let

$G_0 = X;$

$G_1 = X - \{x_1\} = \{x_2, x_3, \cdots, x_n\};$

$G_2 = X - \{x_1, x_2\} = \{x_3, x_4, \cdots, x_n\};$

$G_3 = X - \{x_1, x_2, x_3\} = \{x_4, x_5, \cdots, x_n\};$

$\vdots$

$G_k = X - \{x_1, x_2, \cdots, x_k\} = \{x_{k+1}, x_{k+2}, \cdots, x_n\},$

$1 \leq k \leq n$. Then $\tau_k = \{\emptyset, G_t\}_{t=0}^{k}$ is a topology on X finer than τ_{k-1}. Hence $\{\tau_k\}_{k=1}^{n}$ is a (finite) family of topologies on X forming a chain in that

$$\tau_1 < \tau_2 < \cdots < \tau_n.$$

We also see that

$$\tau_1^c = \{\emptyset, X, \{x_1\}\};$$

$$\tau_2^c = \{\emptyset, X, \{x_1\}, \{x_1, x_2\}\}, \text{ etc.}$$

are topologies (in chain) on X.
Proof of Theorem 3.1 can be extended to any set|even if in nite|with a chain of complement topologies. The next corollary states this.

Corollary 3.1 *Let $C = \{\tau_\alpha\}_{\alpha \in \Delta}$ be a chain of complement topologies on any set X. Then the family $C^* = \{\tau_\alpha^c : \tau_\alpha \in C\}_{\alpha \in \Delta}$ of complements of the topologies in C is also a chain of complement topologies on X. Conversely, the family of the complements of the topologies in a chain of complement topologies on any set X is itself also a chain of complement topologies on X.*

More Examples

[1] The usual topology u on the set R of real numbers is not closed under arbitrary intersections and is thus not a complement topology.

[2] The usual topology on the Cartesian plane is not closed under arbitrary intersections and is, hence, not a complement topology.

[3] The lower limit (or Sorgenfrey) topology on R is not closed under arbitrary intersections and is also not a complement topology.

[4] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, and let $k \in N$ be such that $2k - 1 \leq n$. Then

$$\tau_{2k-1} = \{\emptyset, X, \{x_1\}, \{x_1, x_3\}, \cdots, \{x_1, x_3, \cdots, x_{2k-1}\}\}$$

is a topology on X, for $1 \leq k \leq \left[\frac{n-1}{2}\right]$. We see also that

$$\tau_{2k-1}^c = \{X, \emptyset, \{x_2, \cdots, x_n\}, \{x_2, x_4, \cdots, x_n\}, \cdots, \{x_2, x_4, \cdots, x_n\}\}$$

is a topology on X.

[5] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, and let $k \in N$ be such that $2k - 1 \leq n$. Then $\tau_{2k-1} = \left\{\emptyset, X, \bigcup_{t=1}^{k}\{x_{2t-1}\}\right\}$ is a topology on X, for $1 \leq k \leq [\frac{n-1}{2}]$. Also $\tau_{2k-1}^c = \left\{X, \emptyset, X - \bigcup_{t=1}^{k}\{x_{2t-1}\}\right\}$ is a topology on X; $1 \leq k \leq [\frac{n-1}{2}]$.

[6] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, $n = mt + r, 0 \leq r < m$. Let $\tau_m = \{\emptyset, X, \{x_m\}, \{x_m, x_{2m}\}, \cdots, \{x_m, x_{2m}, \cdots, x_{tm}\}\} = \left\{\emptyset, X, \bigcup_{i=1}^{k}\{x_{im}\}\right\}; 1 \leq k \leq t$. Then τ_m is a topology on X. And we see that $\left\{X, \emptyset, X - \bigcup_{i=1}^{k}\{x_{im}\}\right\}; 1 \leq k \leq t$ is a topology on X.

Remark

It is known that a topological space (X, τ) is a T_1-space if, and only if, singletons are τ-closed subsets of X. It is observable from the foregoing that if a topology τ on a set X is a complement topology then the very sets which are seen as τ-closed are the sets which constitute the open sets of another topology on X, with equal cardinality as τ. These imply the following.

Corollary 3.2 *If (X, τ) is a T_1 topological space, then τ is a complement topology if, and only if, τ is the discrete topology of X.*

Proof:

Since (X, τ) is T_1, singletons of X are τ-closed. Since τ is a complement topology on X and singletons of X are τ-closed, it follows that singletons are among the τ^c-open sets. Hence every subset of X is τ^c-open, implying that τ^c is the discrete topology of X. Since $(\tau^c)^c = \tau$, it follows that τ is the discrete topology of X.

Remark

That a topology is a complement topology does not imply that it is T_1. Also, every T_1-space is not a complement topology. By Corollary 3.2, a T_1-space which is a complement topology must be a discrete topology. It follows that if a T_1-space is not discrete then it cannot be a complement topology. For example, the set R of real numbers with its usual topology u is T_1 but u is not a complement topology. Hence all complement topologies are not T_1 and all T_1-spaces are not complement topologies.

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Chika Moore. "Complement Topologies." IOSR Journal of Mathematics (IOSR-JM) 13.3 (2017): 73-77.