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Inverse Problem for Singular Sturm-Liouville operator

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***Abstract**: In this article, the inverse problem for the singular Sturm-Liouville operator is studied. A set of values of eigenfunctions in some internal point and spectrum are given as a data. Uniqueness theorems are also obtained.*

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I. Introduction

Inverse spectral analysis has been an important research topic in mathematical physics. Inverse problems of spectral analysis involve reconstruction of a linear operator from its spectral characteristics [1,3-9]. For inverse Sturm-Liouville problems, such characteristics are two spectra for different boundary condiyions, one spectrum and normalizing constants, spectral functions, scattering data, Weyl function [1,3-18]. An early important result in this direction, which gave vital imputes for the further development of inverse problem theory, was obtained in [2]. Inverse problem for interior spectral data of the differential operator lies in reconstructing this operator by some eigenvalues and information on eigenfunctions at some an internal point in the interval considered. The similar problems for the Sturm-Liouville and diffusion operator was studied in [19–25].
Consider the following singular Sturm-Liouville operator L satisfying (1)-(3)

$$Ly = -y'' + \left[q(x) - \frac{1}{4\sin^2 x} \right] y = \lambda y \qquad (1)$$

with boundary conditions

$$y(0) = 0 \qquad (2)$$

$$y'(\pi - \varepsilon, \lambda) + Hy(\pi - \varepsilon, \lambda), \qquad (3)$$

where $q(x)$ is summable, ε any positive number, H finite real number and λ spectral parameter. The operator L is self adjoint on the $L_2(\pi, 0)$ and with (2), (3) boundary conditions has a discrete spectrum $\{\lambda_n\}$.

Let us introduce the second singular Sturm-Liouville operator L satisfying

$$\tilde{L}y = -y'' + \left[\tilde{q}(x) - \frac{1}{4\sin^2 x} \right] y = \lambda y \qquad (4)$$

subject to the same boundary conditions (2), (3) where $\tilde{q}(x)$ is summable. The operator $\tilde{L}$ is self adjoint on the $L_2(\pi, 0)$ and with (2), (3) boundary conditions has a discrete spectrum $\{\tilde{\lambda}_n\}$.

II. Main Results

The Legendre equation is

$$(1 - t^2)y'' - 2ty' + n(n+1)y = 0.$$

First, we let

$$t = \cos x, \quad z = y(t) \quad and \quad y = z\sqrt{\sin x}$$

then y satisfies

$$-y'' - \left[(\lambda + \tfrac{1}{4}) + \frac{1}{4\sin^2 x} \right] y = \lambda y$$

where $\lambda = n(n+1)$. For $0 < \varepsilon < x \le \pi - \varepsilon < \pi$ and n sufficiently large, we conclude that Legendre functions are [18],

$$P_n(x) \approx \sqrt{\tfrac{2}{\pi n}}\cos\left[\left(n+\tfrac{1}{2}\right)x-\tfrac{\pi}{4}\right]$$

Solution of the equation (1)-(3) satisfying $\varphi(0,\lambda)=0$, $\varphi'(0,\lambda)=1$ boundary conditions is

$$\varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}\,x}\int_0^x \sin\sqrt{\lambda}\,(x-t)\left[q(t)-\tfrac{1}{4\sin^2 t}\right]\varphi(t,\lambda)\,dt \qquad (5)$$

Eigenvalues of the problem (1)-(3) are the roots of the (3). These spectral characteristics and eigenfunctions satisfy the following asymptotic expression, respectively

$$\rho_n=\sqrt{\lambda_n}=n+\tfrac{1}{2}+\frac{c}{\pi\left(n+\frac{1}{2}\right)}+o\left(\tfrac{1}{n^2}\right), \qquad (6)$$

$$\varphi(x,\lambda_n)=\frac{1}{n+\frac{1}{2}}\sin\left(n+\tfrac{1}{2}\right)x+o\left(\tfrac{1}{n^2}\right), \qquad (7)$$

where $c=\frac{1}{\pi}\left(H+\frac{1}{2}\int_0^\pi q(t)\,dt\right)$ [18].

When $b=\frac{\pi}{2}$ we get the following uniqueness Theorem 2.1

Theorem 2.1 *If for every n ∈N we have*

$$\lambda_n=\tilde{\lambda}_n, \quad \frac{y'_n\left(\frac{\pi}{2}\right)}{y_n\left(\frac{\pi}{2}\right)}=\frac{\tilde{y}'_n\left(\frac{\pi}{2}\right)}{\tilde{y}_n\left(\frac{\pi}{2}\right)} \qquad (8)$$

then

$q(x)=\tilde{q}(x)$ a.e on the interval $(0,\pi)$.

In the case $b\neq\frac{\pi}{2}$ the uniqueness of $q(x)$ can be proved if we require the knowledge of a part of the second spectrum.

Let $m(n)$ be a sequence of natural numbers with a property

$$m(n)=\tfrac{n}{\sigma}(1+\varepsilon_n),\ \ 0<\sigma\le 1,\ \ \varepsilon_n\to 0. \qquad (9)$$

Lemma 2.1 *Let m(n) be a sequence of natural numbers satisfying (9) and* $b\in\left(0,\frac{\pi}{2}\right)$ *are so chosen that* $\sigma>\frac{2b}{\pi}$ *. If for any n ∈N*

$$\lambda_{m(n)}=\tilde{\lambda}_{m(n)}, \quad \frac{y'_{m(n)}(b)}{y_{m(n)}(b)}=\frac{\tilde{y}'_{m(n)}(b)}{\tilde{y}_{m(n)}(b)} \qquad (10)$$

then $q(x)=\tilde{q}(x)$ *a.e on* $(0,b]$

Let $l(n)$ and $r(n)$ be a sequence of natural numbers such that

$$l(n)=\tfrac{n}{\sigma_1}(1+\varepsilon_{1,n}),\ \ 0<\sigma_1\le 1,\ \varepsilon_{1,n}\to 0 \qquad (11)$$

$$r(n)=\tfrac{n}{\sigma_2}(1+\varepsilon_{2,n}),\ \ 0<\sigma_2\le 1,\ \varepsilon_{2,n}\to 0 \qquad (12)$$

and let μ_n be the eigenvalues of the problem (1), (2) and (13) and $\tilde{\mu}_n$ be the eigenvalues of the problem (4), (2) and (13)

$$y'(\pi-\varepsilon,\lambda)+\tilde{H}y(\pi-\varepsilon,\lambda)=0 \qquad (13)$$

Using Mochizuki and Trooshin's method from Lemma 2.1 and Theorem 2.1, we will prove that the following Theorem 2.2 holds.

Theorem 2.2 *Let l(n) and r(n) be a sequence of natural numbers satisfying (11) and (12), and* $\frac{\pi}{2} < b < \pi$ *are so chosen that* $\sigma_1 > \frac{2b}{\pi} - 1, \quad \sigma_2 > 2 - \frac{2b}{\pi}$ *If for any* $n \in N$ *we have*

$$\lambda_n = \tilde{\lambda}_n, \mu_{l(n)} = \tilde{\mu}_{l(n)} \quad and \quad \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} = \frac{\tilde{y}'_{r(n)}(b)}{\tilde{y}_{r(n)}(b)} \tag{14}$$

then

$$q(x) = \tilde{q}(x) \text{ a.e on } (0, \pi).$$

III. Proof of the Main Results

Proof of Theorem 2.1 Error! Reference source not found. Before proving the Theorem 2.1,we will mention some results, which will be needed later. We get the initial value problems

$$-y'' + \left[q(x) - \frac{1}{4\sin^2 x}\right] y = \lambda y, \tag{15}$$

$$y(0) = 0, \tag{16}$$

and

$$-\tilde{y}'' + \left[\tilde{q}(x) - \frac{1}{4\sin^2 x}\right] \tilde{y} = \lambda \tilde{y}, \tag{17}$$

$$\tilde{y}(0) = 0 \tag{18}$$

It can be shown that there exists a kernel $K(x,t)$ $\left(\tilde{K}(x,t)\right)$ continuous on $(0,\pi) \text{ x} (0,\pi)$ such that by using the transformation operator every solution of equations (15), (16) and (17), (18) can be expressed in the form

$$y(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x K(x,t)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}dt + o\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right), \tag{19}$$

$$\tilde{y}(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \tilde{K}(x,t)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}dt + o\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right), \tag{20}$$

respectively, where $\tau = \left|Im\sqrt{\lambda}\right|$and the kernel $K(x,t)$ $\left(\tilde{K}(x,t)\right)$ is the solution of the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2} + \left(\frac{1}{4\sin^2 x} - q(x)\right)K(x,t) = \frac{\partial^2 K(x,t)}{\partial t^2} + \frac{1}{4\sin^2 x}K(x,t)$$

subject to the boundary conditions

$$2\frac{dK(x,x)}{dx} = q(x),$$

$$K(x,0) = 0$$

Multiplying (15) by $\tilde{y}(x,\lambda)$ and (17) by $y(x,\lambda)$, subtracting and integrating from 0 to $\frac{\pi}{2}$ we obtain

$$\int_0^{\frac{\pi}{2}} \left(q(x) - \tilde{q}(x)\right) y(x,\lambda)\tilde{y}(x,\lambda)dx = \left(\tilde{y}(x,\lambda)y'(x,\lambda) - y(x,\lambda)\tilde{y}'(x,\lambda)\right)\Big|_0^{\frac{\pi}{2}} \tag{21}$$

The functions $y(x,\lambda)$ and $\tilde{y}(x,\lambda)$ satisfy the same initial conditions (16) and (18), i.e.,

$$\tilde{y}(0,\lambda)y'(0,\lambda) - y(0,\lambda)\tilde{y}'(0,\lambda) = 0$$

Let

$$\square(x) = q(x) - \tilde{q}(x), \tag{22}$$

$$H(\lambda) = \int_0^{\frac{1}{2}} \square(x) y(x,\lambda)\tilde{y}(x,\lambda)dx \tag{23}$$

If the properties of $y(x,\lambda)$ and $\tilde{y}(x,\lambda)$ are considered, the function $H(\lambda)$ is an entire function.

Therefore the condition of the Theorem 2.1 imply,

$$\tilde{y}(\tfrac{\pi}{2},\lambda_n)y'(\tfrac{\pi}{2},\lambda_n) - y(\tfrac{\pi}{2},\lambda_n)\tilde{y}'(\tfrac{\pi}{2},\lambda_n) = 0$$

and hence

$$H(\lambda_n) = 0, \quad n \in N.$$

In addition, using (19) and (23) for $0 < x < \pi$,

$$\left|H(\lambda)\right| \le M \tfrac{\pi}{2}\left|b\right|, \quad b = \operatorname{Im}\left|\sqrt{x}\right|, \tag{24}$$

where M is constant.

Introduce the function

$$\omega(\lambda) = y'(\pi-\varepsilon,\lambda) + Hy(\pi-\varepsilon,\lambda). \tag{25}$$

The zeros of $\omega(\lambda)$ are the eigenvalues of L and hence it has only simple zeros λ_n because of the seperated boundary conditions. By using the asymptotic forms of $y(x,\lambda)$ and $y'(x,\lambda)$, we obtain

$$\omega(\lambda) = \cos\left(n+\tfrac{1}{2}\right)\pi + o(\tfrac{1}{n}). \tag{26}$$

From 26, $\omega(\lambda)$ is an entire function of order $\frac{1}{2}$ of λ. Since the set of zeros of the entire function $\omega(\lambda)$ is contained in the set of zeros $H(\lambda)$ we see that the function

$$\psi(\lambda) = \frac{H(\lambda)}{\omega(\lambda)} \tag{27}$$

is an entire function. From (24) (26) and (27), we get

$$\left|\psi(\lambda)\right| = o(\tfrac{1}{\sqrt{\lambda}})$$

So, for all λ, from the Liouville theorem,

$$\psi(\lambda_n) = 0$$

or

$$H(\lambda) = 0.$$

It was proved in[18] that there exists absolutely continuous function $\tilde{\tilde{K}}(x,t)$ such that we have

$$y(x,\lambda)\tilde{y}(x,\lambda) = \tfrac{1}{2\lambda}\left[1-\cos(2\sqrt{\lambda}x) + \int_0^x \tilde{\tilde{K}}^{\wedge}(x,t)\cos(2\sqrt{\lambda}t)dt\right] + o(\tfrac{e^{rx}}{\sqrt{\lambda}}) \tag{28}$$

where

$$\tilde{\tilde{K}}(x,t) = 2\left[K(x,x-2t) + \tilde{K}(x,x-2t)\right] + 2\left[\int_{-x+2t}^{x} K(x,s)\tilde{K}(x,s-2t)ds + \int_{-x}^{x-2t} K(x,s)\tilde{K}(x,s-2t)ds\right].$$

We are now going to show that $Q(x) = 0$ a.e. on $(0,\frac{\pi}{2})$ From (23) and (28), we have

$$\int_0^{\frac{\pi}{2}} Q(x)\left[1-\cos(2\sqrt{\lambda}x) + \int_0^x \tilde{\tilde{K}}(x,t)\cos(2\sqrt{\lambda}t)dt\right] + o(\frac{e^{rx}}{\sqrt{\lambda}}) = 0$$

This can be written as

$$\int_0^{\frac{\pi}{2}} Q(x)dx - \int_0^{\frac{\pi}{2}} \cos(2\sqrt{\lambda}x)\left[Q(t) + \int_t^{\frac{\pi}{2}} Q(x)\tilde{\tilde{K}}(x,t)dx\right]dt + o(\frac{e^{rx}}{\sqrt{\lambda}}) = 0$$

Let $\lambda \to \infty$ along the real axis, By the Riemann-Lebesgue lemma, we should have

$$\int_0^{\frac{\pi}{2}} Q(x)dx = 0 \tag{29}$$

and

$$\int_0^{\frac{\pi}{2}} \cos(2\sqrt{\lambda}x)\left[Q(t) + \int_t^{\frac{\pi}{2}} Q(x)\tilde{\tilde{K}}(x,t)dx\right]dt = 0 \tag{30}$$

Thus from the completeness of the functions $\cos(2\sqrt{\lambda}t)$ it follows that

$$Q(\tau) + \int_0^{\frac{\pi}{2}} Q(x)\tilde{\tilde{K}}(x,t)dx = 0, \qquad 0 < t < \tfrac{\pi}{2} \tag{31}$$

Thus we have obtained

$$Q(x) = q(x) - \tilde{q}(x) = 0$$

or

$$\tilde{q}(x) = q(x)$$

almost everywhere on $(0,\frac{\pi}{2}]$

To prove that $q(x) = 0$ on $[\frac{\pi}{2},\pi)$ almost everywhere, we should repeat arguments for the supplementary problem

$$Ly = -y'' + \left[q(\pi - x) - \frac{1}{4\sin^2 x}\right]y = \lambda y, \qquad 0 < x < \pi$$

subject to the boundary conditions

$$y(\pi - \varepsilon, \lambda) = 0$$

$$y'(\lambda, 0) + Hy(0,\lambda) = 0$$

Consequently

$$q(x) = \tilde{q}(x) \quad \text{a.e on the interval } (0,\pi)$$

Therefore Theorem 2.1 is proved.

Proof of Lemma 2.1 . As in the proof of Theorem 2.1 we can show that

$$G(\rho) = \int_0^b Q(x)y(x,\lambda)\tilde{y}(x,\lambda)dx = \left(\tilde{y}(x,\lambda)y'(x,\lambda) - y(x,\lambda)\tilde{y}(x,\lambda)\right)\Big|_{x=b} \tag{32}$$

where $\rho = \sqrt{\lambda}$ and $Q(x) = \tilde{q}(x) - q(x)$. From the assumption

$$\frac{y'_{m(n)}(b)}{y_{m(n)}(b)} = \frac{\tilde{y}'_{m(n)}(b)}{\tilde{y}_{m(n)}(b)}$$

together with the initial condition at 0 it follows that,

$$G(\rho_{m(n)}) = 0, \quad n \in N$$

Next, we will show that $G(\rho) = 0$ on the whole ρ plane.

From (32), we see that the entire function G(ρ) is a function of exponential type ≤2b. One has

$$\left|G(\rho)\right| \le Me^{2br\left|sin\theta\right|} \tag{33}$$

where M is positive constant, $\rho = \sqrt{\lambda} = re^{i\theta}$.

Define the indicator of function G(ρ) by;

$$h(\theta) = \lim_{r\to\infty}\sup\frac{\ln\left|G(re^{i\theta})\right|}{r} \tag{34}$$

Since $\left|\mathrm{Im}\sqrt{\lambda}\right| = r\left|\sin\theta\right|$, $\theta = \arg\sqrt{\lambda}$ from (33) and (34) it follows that

$$h(\theta) \le 2b\left|sin\theta\right|. \tag{35}$$

According to the [26] set of zeros of every entire function of the exponential type, not identically zero, satisfies the inequality:

$$\lim_{r\to\infty}\inf\frac{n(r)}{r} \le \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta \tag{36}$$

where $n(r)$ is the number of zeros of G(ρ) in the disk $\left|\rho\right| \le r$. By (35),

$$\frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta \le \frac{b}{\pi}\int_0^{2\pi}\left|\sin\theta\right|d\theta = \frac{4b}{\pi}$$

From the assumption and the known asymptotic expression (6) of the eigenvalues $\sqrt{\lambda_n}$ we obtain;

$$n(r) \ge \sum_{\frac{n}{\sigma}\left[1+o\left(\frac{1}{n}\right)\right]<r} 2\sigma r\left(1+o(1)\right)$$

$r \to \infty$.

For the case $\sigma > \frac{2b}{\pi}$

$$\lim_{r\to\infty}\frac{n(r)}{r} \ge 2\sigma > \frac{4b}{\pi} = 2b\int_0^{2\pi}\left|\sin\theta\right|d\theta \ge \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta \tag{37}$$

The inequalities (36) and (37) imply that $G(r) = 0$ on the whole ρ plane.

Similar to the proof of the Theorem 2.1 we have

$$q(x) = \tilde{q}(x) \quad \text{a.e on the interval } (0,b)$$

This completes the proof of Lemma 2.1.
Proof of Theorem 2.2. From

$$\lambda_{r(n)} = \tilde{\lambda}_{r(n)}, \quad \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} = \frac{\tilde{y}'_{r(n)}(b)}{\tilde{y}_{r(n)}(b)}$$

where $r(n)$ satisfies (12) and $\sigma_2 > 2 - \frac{2b}{\pi}$ according to Lemma 2.1 we get

$$q(x) = \tilde{q}(x) \quad \text{a.e on } [b,\pi) \quad . \tag{38}$$

Thus, it needs to be proved that $q(x) = \tilde{q}(x)$ *a.e on* $(0,b]$ The eigenfunctions $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ satisfy the same boundary condition at π. It means that

$$y_n(x,\lambda_n) = \xi_n \tilde{y}_n(x,\lambda_n) \tag{39}$$

on (b,π) for any $n \in N$ where ξ_n are constants.

From (32) and (39), we get that;

$$G(\rho) = 0, \quad \text{for} \quad \rho^2 = \lambda_n, \quad n \in N,$$

and

$$G(\rho) = 0, \quad \text{for} \quad \rho^2 = \mu_{l(n)}, \quad n \in N,$$

where λ_n and $\mu_{l(n)}$ satisfy (6).

Counting the number of λ_n and $\mu_{l(n)}$ located inside the disc of radius r, we have

$$1 + 2r\left[1 + o(\tfrac{1}{n})\right]$$

of λ_n's and

$$1 + 2r\sigma_1\left[1 + o(\tfrac{1}{n})\right]$$

of $\mu_{l(n)}$'s.

This means that

$$n(r) = 2 + 2\left[r(\sigma_1 + 1) + o(\tfrac{1}{n})\right]$$

and

$$\lim_{r\to\infty} \frac{n(r)}{r} = 2(\sigma_1 + 1)$$

Repeating the last part of the proof of Lemma 2.1, and considering the condition $\sigma_1 > \frac{2b}{\pi} - 1$, we can demonstrate that $G(\rho) = 0$ identically on the whole ρ-plane which implies that

$$q(x) = \tilde{q}(x) \quad \text{a.e on } (0,b]$$

and consequently

$$q(x) = \tilde{q}(x) \quad \text{a.e on } (0,\pi]$$

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