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A Necessary and Sufficient Condition for Linear Systems to Be Observable

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Abstract: *Control system is said to be observable if it contains or is coupled to enough information which enables us to determine precisely the relationship between the input and the resulting output variables at any given finite time. Here, we develop certain necessary and sufficient conditions which assure that the following linear control system with input $x\epsilon E^m$, output $y\epsilon E^p$ and control $u\epsilon E^m$ given by*

$$\dot{x} = Ax(t) + Bu(t), t \geq 0$$
$$y(t) = Hx(t)$$

where A, B and H are real n x n, n x m, and p x n matrices, is conclusively observable.
Keywords*: Characteristic polynomial, controllability, observability.*

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I. Introduction

In the setting E^n, the n-dimensional space, let us examine the linear control system.

$$\left.\begin{aligned} \dot{x} = Ax(t) + Bu(t), t \geq 0 \\ x(0) = x_0 \end{aligned}\right\} \quad (1.1)$$

in which $x\epsilon E^n$, A, B and H are n x n, n x m and p x n constant matrices and u is an m-vector valued measurable function whose values u(t) belong to compact, convex set $\Omega\ of E^m$, the m-dimensional Euclidean space. Such control function u(t) is said to be admissible. In this paper, we assume that $|u_j| \leq 1$, j = 1, 2, …, m. This means that u belongs to some unit cube in E^m. The absolute solution to (1.1) will be designated by x(t, u).

In this work, we are studying the observability of control system (1.1). It is known [1] that for a given dynamical systems, its controllability and its observability are related by some duality principles. So, it is therefore necessary that we discuss these two concepts side by side.

Assume that $y\epsilon E^P$ is the output of (1.1) brought about by the action of some control action u. Then, the system, namely

$$\dot{x} = A(t) + Bu(t) \quad (1.2)$$

and

$$y(t) = Hx(t) \quad (1.3)$$

give the difference between the input variable x and the output variable y(t). In (1.3), $y\epsilon E^p$, and $H\epsilon E^{p\times n}$, which means that H is a constant p x n output matrix.

Generally, the equation (1.2) is called the controllability equation and (1.3) is known as observability equation. The two equations (1.2) and (1.3) are said to be observable if in (1.3) the output y contains enough information that will obviously enable us to estimate the input x in any finite time. May be, because of some far-reaching duality relationship between controllability and observability that made some researchers to concentrate more on controllability for example [2], [3], [4] and others, who gave little attention to observability. It is mainly Lee and Markus [5] and also Hautus [1] who did some considerable research on observability. May be as a result of said duality relationship which exist between the two concepts, which we shall see later, that each of the above researchers throughout his or her considered observability from the framework of controllability.

Here, we intend to establish observability conditions independently of the controllability of the associated dual system. This is what we consider our contribution in this direction. Thus, we are hoping that in a Theorem below which leads us to the main theorem, we hope to see a condition which will help us to generate almost an infinite number of observable systems from a simple observable system. Let us assume some facts which we shall need to achieve our current assignment.

II. Notations and Preliminaries

Here we want to settle a few crucial definitions which will helps understand the work very well. We also want to state vital facts and theorem which form vital pivots to the problem under study. So, we have;

Definition 2.1

A control function $u\epsilon E^m$ is said to be admissible if it is piceevice continuous, differentiable and lies in some closed bounded set.

Definition 2.2

The control system (1.1) is said to be Euclidean controllable if for each $x_0\epsilon E^n$ and each $x_1\epsilon E^n$ there exists a finite time $t_1 \geq 0$ and an admissible control u such that the solution x(t, u) = x(t), say, of (1.1) satisfies $x(0) = x_0$ and $x(t_1) = x_1$.

Note that the system (1.1) is null-controllable if in definition 2.2 above $x_1 = \bar{0}$.

Definition 2.3

The joint system (1.2) and (1.3) is said to be observable if for each initial point $x(0) = x_0$ and any arbitrary time t_0, there exists a finite time t_1, with $t_0 \leq t \leq t_1 < \infty$ such that for each admissible control u and output y(t), $t_0 \leq t \leq t_1$, we can find the input x(t).

We shall also use Hamilton-Cayley Theorem in our investigation, so, we state the theorem thus;

Theorem 2.1 [6]

If A is a square matrix and f(x) is its characteristic polynomial, then f (A) = 0.

We see, this theorem is saying that every square matrix is a root of its characteristic polynomial. Now, we want to state a theorem which gives a computable condition for a linear process to be observable. So, we have;

Theorem 2.2 Observability [5]

The joint system (1.2) and (1.3) which is the linear systems, where $x\epsilon E^n$, $y\epsilon E^p$, $u\epsilon E^m$ and A, B, H are constant matrics of appropriate dimensions, is observable if and only if the rank of U = n, where $U = [H \;\; HA \;\; AH^2 \; \ldots \; HA^{n-1}]^T$ is called an observability matrix, $(.)^T$ denotes transpose.

Nent is the theorem on duality relationship, a famous concept existing between controllability and obervability on which most researchers hinge their study of observability.

Theorem 2.3 [5]. Duality principle.

The control system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Hx \end{aligned}\right\} \qquad (2.1)$$

is controllable if and only if the following dual system.

$$\left.\begin{aligned} \dot{x} &= -A^T x + H^T u \\ y &= B^T x \end{aligned}\right\} \qquad (2.2)$$

where $x\epsilon E^n, y\epsilon E^p$, $u\epsilon E^m$, $A\epsilon E^{n\times n}$, $B\epsilon E^{n\times n}$, $H\epsilon E^{p\times m}$ and $(.)^T$ denotes matrix transpose as applicable, is observable.

We note that the theorem we are going to state next unites Theorems 2.2 and 2.3 together and provides a detailed proof for observability by establishing the controllability of the associated dual system by classical method. So, we have

Theorem 2.4 [5]

The autonomous linear observed process in E^n

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Hx \end{aligned}\right\} \qquad (2.3)$$

is observable if and only if the dual dynamical system.

$$\left.\begin{aligned} \dot{x} &= -A^T x + H^T u \\ y &= B^T x \end{aligned}\right\} \qquad (2.4)$$

is controllable. This holds if and only if

Rank $[H^T \; A^T H^T \; H^{T^2} \; H^T \; \ldots \; A^{T^{n-1}} \; H^T] = n$

We note that the proofs of the above theorems are given in the cited references. So, we are now ready to establish the main result of this work.

III. Main Result

We now state the main result of this paper as;

Theorem 3.1 (Observability Theorem)

In E^n, let us consider linear observed system.

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), t \geq 0 \\ y(t) &= Hx(t) \end{aligned}\right\} \qquad (3.1)$$

where $x\epsilon E^n$, $y\epsilon E^p$, $u\epsilon E^m$ is an admissible control function and A, B, and H are respectively n x n, n x m and p x n constant matrices. If λ is an arbitrary eigenvalues of the constant matrix A, then a necessary and sufficient condition that the linear system (3.1) be observable is that for each vector $x\epsilon E^n$, we have

$$Ax = \lambda x, Hx = 0, implies\ x = 0.$$

Proof

Necessity: We want to show that if the system (3.1) is observable, then for $x\epsilon E^n$ with $Ax = \lambda x, Hx = 0$ implies x = 0.

Now, suppose that there exists some vector $x\epsilon E^n$ such that $Ax = \lambda x, Hx = 0$ but x ≠ 0. Now, $Ax = \lambda x, Hx = 0$ implies that

$$HA^k x = H\lambda^k x \quad (k = 1, 2, \dots, n-1).$$
$$= \lambda^k Hx$$

= 0

(since Hx = 0 by hypothesis)

This applies if and only if for k = 1, 2, …, n – 1, the columns of the matrix $[H\ \ HA\ \ HA^2 \dots Ha^{n-1}]^T$ are linearly independent; that is

$$Rank \begin{bmatrix} H \\ HA \\ HA^2 \\ . \\ . \\ . \\ HA^{n-1} \end{bmatrix} < n$$

contradicting theorem 2.2 for observability.

Hence, $Ax = \lambda x, Hx = 0\ with\ x \neq 0$ leads to the conclusion that the given system (3.1) is not observable. Hence, we have proved that if the system (3.1) is observable then $Ax = \lambda x, Hx = 0$ implies that $x = 0$ (3.2) for each x.

Sufficiency:

Now, we assume that the system (3.1) is not observable. Then, Theorem 2.2 implies that

$$Rank \begin{bmatrix} H \\ HA \\ \vdots \\ . \\ HA^{n-1} \end{bmatrix} < n$$

Hence, for some non-zero scalar α ≠ 0, we have

$\alpha HA^k = 0$

This implies that

$H\alpha A^k = 0\ (k = 0, 1, 2, \dots, n-1).$ (3.3)

Let $\bar{\emptyset}(x)$ be a polynomial of degree n_1, say, with $1 \leq n_1 < n$ such that

$\alpha\bar{\phi}(A) = 0$ (3.4)

The existence of such a polynomial is clearly in complete alliance with Theorem 2.1 (i.e. Hamilton – Cayley Theorem). Moreover, for λ and some polynomial $\bar{\phi}(x)$ of degree n – 1, we can define $\bar{\phi}(x) = \psi(x)(x - \lambda)$, (3.5)

Also, for $\eta \neq 0$, some suitable non-zero vector, let us define

$\eta = \alpha\psi(A)$ (3.6)

Then from (3.4), (3.5) and (3.6), we get that

$\alpha\bar{\phi}(A) = \alpha\psi(A)(A - \lambda I)$, I an identity matrix.

$= \eta(A - \lambda I)$

= 0

and from this, we get that

$(A - \lambda I)\eta = 0, \eta \neq 0$ (3.7)

Also from (3.3) and (3.6) we get

$H\eta = 0, \eta \neq 0$ (3.8)

In (3.7) and (3.8) above, we get, without loss of generality, if we set $\eta = x$, we easily see from these equations that $Ax = \lambda x, Hx = 0 \Longrightarrow x \neq 0$. Hence if the system (3.1) is not observable, we infer that $Ax = \lambda x, Hx = 0, x \neq 0$ and so we have proved that $Ax = \lambda x, Hx = 0 \Longrightarrow x = 0$ implies that the given system (3.1) is observable, and so completing the proof of the theorem.

The next theorem, which is an important fallout from Theorem 3.1 is a method of generating observable systems from any give linear observable system. So, we have;

Theorem 3.2

If the linear observed process, (3.1) is observable, then so is the control system defined by

$$\left.\begin{aligned} \dot{x}(t) &= Mx(t) + Bu(t),\ t \geq 0 \\ y &= Hx \end{aligned}\right\} \tag{3.9}$$

where x, y, u, A, B and H are as defined before in theorem 3.1, provided M is an nxn constant matrix defined by

$$M = A + NH \tag{3.10}$$

where N is any nxp constant matrix.

Proof:

For λ some eigenvalue of A, the observed process (3.9) will be observable if and only if $Mx = \lambda x, Hx = 0 \Longrightarrow x = 0\ for\ x\epsilon E^n$. But, this is true since

$$\begin{aligned} Mx &= (A + NH)x \\ &= Ax + NHx \\ &= \lambda x (since\ Ax = \lambda x, Hx = 0 \Longrightarrow x = 0 \end{aligned}$$

as (3.1) is, by hypotheses, observable) and $Hx = 0 \Longrightarrow x = 0$ (by hypothesis)

Hence, we have shown that $Mx = \lambda x,\ Hx = 0 \Longrightarrow x = 0$ and the given system (3.9) is observable. Hence, the Theorem is proved.

In the next part, we given an example to illustrate the application of the main result of this paper.

1. Example;

Consider in E^2, the following linear observed system

$$\left.\begin{aligned} \ddot{x} - x = 2u,\ \ |u| \leq 1 \\ y = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} x \end{aligned}\right\} \tag{4.1}$$

for some admissible control u.

So,

Let $\dot{x} = \dot{x}_1 = x_2$

Then, we have $\dot{x}_2 = x_1 + 2u$.

So that the system (4.1) assumes the standard form

$$\left.\begin{aligned} \dot{x} &= Ax + Bx \\ y &= Ax \end{aligned}\right\} \tag{4.2}$$

provided.

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}\ and\ H = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$.

Clearly the eigen values of A are $\lambda = 1, -1$. Now, let us choose $\lambda = 1$ (one can make use of an alternative choice of $\lambda = -1$ to get the same conclusion)

Suppose we select

$x = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \epsilon E^2$ say, to be an arbitrary element in E^2. Then

$Ax = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_2 \\ \varepsilon_1 \end{pmatrix} and\ \lambda x = 1. \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$

If $Ax = \lambda x$ we obtain

$\begin{pmatrix} \varepsilon_2 \\ \varepsilon_1 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$, we get

$$\varepsilon_1 = \varepsilon_2 \tag{4.3}$$

Also $Hx = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{bmatrix} \varepsilon_1 - \varepsilon_2 \\ \varepsilon_2 \\ 2\varepsilon_1 + \varepsilon_2 \end{bmatrix}$.

and if $Hx = 0$, we have

$\begin{bmatrix} \varepsilon_1 - \varepsilon_2 \\ \varepsilon_2 \\ 2\varepsilon_1 + \varepsilon_2 \end{bmatrix} = 0.$

From this, we get
$$\left.\begin{aligned} \varepsilon_1 &= \varepsilon_2 \\ \varepsilon_2 &= 0 \\ 2\varepsilon_1 &= \varepsilon_2 \end{aligned}\right\} \qquad (4.4)$$

For consistency, the above two systems (4.3) and (4.4) gives $\varepsilon_1 = \varepsilon_2 = 0$. this means that $x = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = 0$

Conclusion:

So, we see that for the above system (4.1) $Ax = \lambda x, Hx = 0 \Longrightarrow x = 0$ for any $x \epsilon E^2$ and so the given system (4.1) is clearly observable.

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