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Brief Summary of Frequently-used Properties of the Floor Function

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Abstract: *The article makes a brief summary on the frequently-used properties of the floor function. The properties include basic inequalities, conditional inequalities and basic equalities that are collected from different publications and are helpful for scholars of mathematics and computer science and technology.*
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I. Introduction

The floor function, which is also called the greatest integer function (see in [1]), is a function that takes an integer value. For arbitrary real number *x*, the floor function of *x*, denoted by $\lfloor x \rfloor$, is defined by an inequality of $x-1<\lfloor x \rfloor \le x$. The floor function frequently occurs in many aspects of mathematics and computer science. However, as stated in article [2], except the Graham's book [3], one can hardly find a general know-of the properties of the floor function though one can find something in the Internet of free wikipedia [4]. Since Graham's book was first published 30 year's ago and its following-up editions made few modification on the part of the floor function, it is necessary to sort out the properties of the function as a reference for researchers. This article summaries briefly the frequently used properties of the floor function so as for reader to have a reference in their studies.

II. Definitions and Notations

The floor function of real number x is denoted by symbol $\lfloor x \rfloor$ that satisfies $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$; the fraction part of x is denoted by symbol $\{x\}$ that satisfies $x = \lfloor x \rfloor + \{x\}$; the ceiling function of *x* is denoted by symbol $\lceil x \rceil$ that fits $x \le \lceil x \rceil < x+1$. In this whole article, $A \Rightarrow B$ means conclusion *B* can be derived from condition *A*; $A \Leftrightarrow B$ means *B* holds if and only if *A* holds. Symbol $\boldsymbol{Z}$ means the integer set, $x \in \boldsymbol{Z}$ means *x* is an integer and $x \notin Z$ indicates *x* is not an integer.

III. Frequently-used Properties of the Floor Function

The following properties of the floor functions are sorted by basic inequalities, conditional inequalities and basic equalities.

3.1 Basic Inequalities

In the following inequalities, *x* and *y* are real numbers by default.

(P1) [1] $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$

(P2) [5] $\lfloor x \rfloor - \lfloor y \rfloor - 1 \le \lfloor x-y \rfloor \le \lfloor x \rfloor - \lfloor y \rfloor < \lfloor x \rfloor - \lfloor y \rfloor + 1$

(P3) [1][3] $\lfloor 2x \rfloor + \lfloor 2y \rfloor \ge \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor$

(P4) [5] $\lfloor (m+n)x \rfloor + \lfloor (m+n)y \rfloor \ge \lfloor mx \rfloor + \lfloor my \rfloor + \lfloor nx+ny \rfloor$ with *m* and *n* being positive integers

(P5) [5] $\lfloor nx \rfloor + \lfloor ny \rfloor \ge (n-1)\lfloor x+y \rfloor + \lfloor x \rfloor + \lfloor y \rfloor$ with *n* being a positive integer

(P6) [1][5] $\lfloor xy \rfloor \ge \lfloor x \rfloor \lfloor y \rfloor$ with $x, y \ge 0$.

(P7) [6] $\left\lfloor \frac{y}{x} \right\rfloor \le \frac{\lfloor y \rfloor}{\lfloor x \rfloor}$ with $x \ge 1$ and $y > 0$.

(P8) [3] $n\lfloor x \rfloor \le \lfloor nx \rfloor$; $n\lfloor x \rfloor = \lfloor nx \rfloor \Leftrightarrow n\{x\} < 1$, where *n* is a positive integer.

(P9) [7] $\left\lfloor \frac{q}{p} \right\rfloor \ge \frac{q+1}{p} - 1$ for arbitrary positive integers *p* and *q*;

3.2 Conditional Inequalities

In the following inequalities, *x* and *y* are real numbers, and *n* is an integer.

(P10)[3] $x < n \Leftrightarrow \lfloor x \rfloor < n$, $n \le x \Leftrightarrow n \le \lfloor x \rfloor$

(P11)[3] $x < n \le y \Leftrightarrow \lfloor x \rfloor < n \le \lfloor y \rfloor$

(P12) [2] $\lfloor x \rfloor > \lfloor y \rfloor \Rightarrow x > y$

(P13) [2][5] $x \le y \Rightarrow \lfloor x \rfloor \le \lfloor y \rfloor$

3.3 Basic Equalities

In the following equalities, *x* and *y* are real numbers, *m* and *n* are integers.

(P14) [3][5] $\lfloor n + x \rfloor = n + \lfloor x \rfloor$.

(P15)[5] $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$ with $m \ge 1$.

(P16)[5] $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor, x \in \boldsymbol{Z} \\ -\lfloor x \rfloor - 1, x \notin \boldsymbol{Z} \end{cases}$

(P17)[3][5] $\lfloor nx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \ldots + \left\lfloor x + \frac{n-1}{n} \right\rfloor$ with *n*>0, particularly, $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor$ and $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor$.

(P18)[3] $\lfloor x \rfloor = \left\lfloor \frac{x}{n} \right\rfloor + \left\lfloor \frac{1+x}{n} \right\rfloor + \ldots + \left\lfloor \frac{n-1+x}{n} \right\rfloor$, particularly, $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor$

(P19)[3] $\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ with $m \ge 1$.

(P20)[1][3] $\lfloor \sqrt{x} \rfloor = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor$

(P21)[3] $\lfloor \log_b x \rfloor = \lfloor \log_b \lfloor x \rfloor \rfloor$

(P22)[3] $\lfloor \log_b m \rfloor + 1 = \lceil \log_b (m+1) \rceil$ with $m \ge 1$.

(P23)[3] $\left\lfloor \frac{\left\lfloor \frac{a}{b} \right\rfloor}{c} \right\rfloor = \left\lfloor \frac{a}{bc} \right\rfloor$ for arbitrary integer *a* and positive integers *b* and *c*.

(P24) [1][5] $\left\lfloor \frac{m+1}{n} \right\rfloor = \begin{cases} \left\lfloor \frac{m}{n} \right\rfloor \quad , n \nmid m+1 \\ \left\lfloor \frac{m}{n} \right\rfloor + 1, n \mid m+1 \end{cases}$

(P25)[5] $\sum_{1 \le n \le x} 1 = \lfloor x \rfloor$

(P26)[7] $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor = \lfloor \sqrt{4n+3} \rfloor$

IV. Applications in Number Theory

The floor function is widely applied in number theory. Here list several very frequently used theorems.

(P27)[3] It needs $\lfloor \log_2 N \rfloor + 1$ binary bits to express decimal integer *N* in its binary expression.

(P28) [9] Let *N* be an integer; then $N - \lfloor \sqrt{N} \rfloor^2 \ge 0$.

(P29)[5]Let *m* and *p* be positive integers; then number of *p*'s multiples from 1 to *m* is calculated by $\left\lfloor \frac{m}{p} \right\rfloor$.

(**P30**)[8]Let m, n and *p* be positive integers such that $1 < p < m < n$; then number of *p*'s multiples from *m* to *n* is calculated by

$$\nu(m,n,p) = \begin{cases} \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor, p \nmid m \\ \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor + 1, p \mid m \end{cases}$$

(**P31**) Arbitrary positive integer *i* yields

$$i - 1 \le 2\left\lfloor \frac{i}{2} \right\rfloor \le i$$

arbitrary positive even integer *e* yields

$$2\left\lfloor \frac{e}{2} \right\rfloor = e$$

and arbitrary positive old integer *o* yields

$$2\left\lfloor \frac{o}{2} \right\rfloor = o - 1$$

Proof. By definition of the floor function, for arbitrary real x , it holds $\frac{x}{2} - 1 < \left\lfloor \frac{x}{2} \right\rfloor \le \frac{x}{2}$, namely, $x - 2 < 2\left\lfloor \frac{x}{2} \right\rfloor \le x$. Hence arbitrary positive integer *i* yields $i - 1 \le 2\left\lfloor \frac{i}{2} \right\rfloor \le i$. When e is even, let $e = 2s$ with $s > 0$; then $2\left\lfloor \frac{e}{2} \right\rfloor = 2\lfloor s \rfloor = 2s = e$. For the case of o being an odd integer, let $o = 2s - 1$ with $s > 0$; then it yields

$$2\left\lfloor \frac{o}{2} \right\rfloor = 2\left\lfloor s - 1 + \frac{1}{2} \right\rfloor = 2(\lfloor 2(s-1) \rfloor - \lfloor s - 1 \rfloor) = 2s - 1 - 1 = o - 1$$

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