IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 6 Ver. I (Nov. - Dec. 2017), PP 33-47
www.iosrjournals.org

Delay Differential Equations Using Market Equilibrium

Subhransu Sundar Das[1], Dhirendra Kumar Dalai[2], Purna Chandra Nayak[3].
[1]*Department of Mathematics, Semiliguda College, Semiliguda, Koraput-764036,Odisha, India*
[2]*Department of Mathematics, Govt. (Jr) Science College, Ayeba, Kendrapara,Odisha, India*
[3]*Department of Mathematics, Bhadrak (Auto) College, Bhadrak, Odisha, India*

Abstract: *This paper is an introduction for solving Delay Differential Equations (DDEs) using market equilibrium. By market we mean the conditions under which producers sell and consumers buy a certain commodity. The term market is used when only one commodity is being bought and sold and the word multimarket is used when more than one commodity is involved. The price, demand and supply of any one good affects the prices etc. of any other goods and vice-versa. DDEs discussed in this paper are linear. The most fundamental Functional Differential Equations(FDEs) is the linear first order Delay Differential Equations (DDEs).Both DDEs and FDEs are used as modeling tools in models in Economics. We discuss the solution of constant coefficient DDEs by the "Method Of Characteristics" (MOC) and we show how to solve more general DDEs using Method Of Steps (MOS). Further the general first order DDEs are considered and Idempotent Differential Equations (IDEs) plays an important role in DDEs. Further every linear Idempotent Differential Equation (IDE) can be solved by differentiation, which is shown in Theorem : 1. Lastly we have shown that there is only one solution to both different systems and the two systems are equivalent to each other.*

Key words: *Delay differential equations, Demand function, Equivalent systems, Market equilibrium, Method of characteristics, Method of steps, Stability of the equilibrium, Supply function.*

Date of Submission: 02-10-2017 Date of acceptance: 16-11-2017

I. Introduction

DDEs are very useful in various fields such as age-structured population growth, control theory, and any model involving responses with non-zero delays.
The general first order DDE has the form :
$y'(t) = f(t, y(t), y(t-\tau))$,for some given $\tau > 0$.
Here , $y'(t)$ depends on the value of y at some time $t-\tau$ in the past, as well as depending upon the current value of y, and other functions of t as determined by f.

1.1 Linear DDEs

The most fundamental Functional Differential Equation (FDE) is the linear first order Delay Differential Equation,
$$y'(t) = a_1(t)y(t) + a_2(t)y(t-\tau), \text{ for } t \geq 0. \quad ...(1)$$
Equation (1) is usually accompanied by an auxiliary condition, stated interms of a function revealing the state of the system for a period prior to the initial time, t = 0

To describe, a complete statement of the first order delay system,
Let $\tau > 0$ and a_1, a_2 be the class of continuously differentiable functions i.e, the class c' functions on $[0,\tau]$ and let $\phi(t)$ be a class c' function on $[-\tau,0]$, then there is a unique function y(t) satisfying the system:
$$y'(t) = a_1(t)y(t) + a_2(t)y(t-\tau), t \in [0,\tau]` \quad ...(2)$$
$$y(t) = \phi(t), \quad t \in [-\tau,0] \quad ...(3)$$
The auxiliary function or the auxiliary condition in equation (3) is called the history function or remote control function, describing the behavior of y(t) on a past time interval other than the one upon which the differential equation is defined.
Unique solution of the system of equations (2) and (3) :
We have
$$y'(t) = a_1(t)y(t) + a_2(t)y(t-\tau), t \in [0,\tau]$$
$$y(t) = \phi(t), \ t \in [-\tau,0]$$

Let u(t) and v(t) are the different solutions of (2) and (3)

$$\therefore \left.\begin{aligned} u'(t) &= a_1(t)u(t) + a_2(t)u(t-\tau) \\ v'(t) &= a_1(t)v(t) + a_2(t)v(t-\tau) \end{aligned}\right\}, t \in [0,\tau]$$

Also,

$$\left.\begin{aligned} u(t) &= \phi(t) \\ v(t) &= \phi(t) \end{aligned}\right\}, t \in [-\tau, 0]$$

Let, $\mathrm{u(t) - v(t) = g(t)}$

$$\Rightarrow \mathrm{g'(t) = u'(t) - v'(t)}$$

$$\Rightarrow g'(t) = a_1(t)\{u(t) - v(t)\} + a_2(t)\{u(t-\tau) - v(t-\tau)\}$$

$$\Rightarrow g'(t) = a_1(t)g(t) + a_2(t)g(t-\tau), t \in [0,\tau]$$

So, g(t) satisfies equation (2),

i.e, $g'(t) = a_1(t)g(t) + a_2(t)g(t-\tau), \quad for \quad t \in [0,\tau]$

Again,

$\mathrm{u(t) - v(t) = 0,}$ for $\mathrm{t \in [-\tau, 0]}$

$\Rightarrow \mathrm{g(t) = 0}$ for $\mathrm{t \in [-\tau, 0]}$

$\Rightarrow \mathrm{g(t-\tau) = 0}$ for $\mathrm{t \in [0, \tau]}$

So, the resulting first order ODE system is

$g'(t) = a_1(t)g(t), \; for \; t \in [0,\tau]$ with $\mathrm{g(0) = 0}$

$\Rightarrow \mathrm{g(t) = 0, \; for \; t \in [0, \tau]}$

So, $\mathrm{g(t) = 0}$ on $[0, \tau]$ as well as on $[-\tau, 0]$

$\Rightarrow \mathrm{g(t) = 0}$ on $[-\tau, \tau]$

$\Rightarrow \mathrm{u(t) - v(t) = 0}$ on $[-\tau, \tau]$

$\Rightarrow \mathrm{u(t) = v(t)}$ on $[-\tau, \tau]$

$\Rightarrow$ the solution of system of equations (2) and (3) is unique.

Generally,

(i) The Method Of Characteristics (MOC) can be used to solve first order DDE with constant coefficients in equation (2) i.e,

$y'(t) = a_1 y(t) + a_2 y(t-\tau)$, where both a_1, a_2 are constants.

Also, the MOC can be applied to a higher order constant coefficients linear DDE.

(ii) The Method Of Steps (MOS) is better to understand and can be used to solve DDEs with variable coefficients. The basic of this method converts the DDE on a given interval to an ODE over that interval, by using the known history function for that interval. The resulting equation is solved and the process is repeated in the next interval with the newly found solution solving as the history function for the next interval.

1.1.1 Solution of general first order linear DDE by using Method Of Characteristics (MOC) :

Method of characteristics (MOC) can be used to solve first order DDE with constant coefficients and the simplest case of equation (2) is

$y'(t) = a_1 y(t) + a_2 y(t-\tau)$, where both a_1, a_2 are constants. …(4)

In the process of MOC, we will use Lambert w-function, W(z), which is the inverse of the equation $Z(w) = we^w$

i.e, $\mathrm{z^{-1} = W}$

i.e, we say when $z(w) = we^w$

$$\Rightarrow w = z^{-1}(we^w)$$

$$\Rightarrow w = W(we^w)$$

Let, $\mathrm{y = ce^{mt}}$, where c is any arbitrary constant and for some constant m(real or complex) be the solution of equation (4).

From equation (4), we get,

$$c\,me^{mt} = a_1 c\,e^{mt} + a_2 c\ e^{m(t-\tau)}$$
$$\Rightarrow cme^{mt} = a_1 c\,e^{mt} + a_2 c\ e^{mt} e^{-m\tau}$$
$$\Rightarrow m = a_1 + a_2 e^{-m\tau}$$
$$\Rightarrow (m - a_1) = \frac{a_2}{e^{m\tau}}$$
$$\Rightarrow (m - a_1)e^{m\tau} = a_2$$
$$\Rightarrow (m - a_1)e^{m\tau} - a_2 = 0 \qquad \ldots(5)$$

Which is the characteristics equation for the DDE in equation (4)

(I) When $a_2 = 0$, then characteristic equation becomes

$$(m - a_1)e^{m\tau} = 0$$
$$\Rightarrow m - a_1 = 0 \text{ as } e^{m\tau} \neq 0$$
$$\Rightarrow m = a_1$$

Thus, $y(t) = ce^{a_1 t}$ is the solution to the ODE $y'(t) = a_1 y(t)$, which is equation (4) with $a_2 = 0$.

(II) When $a_1 = 0$, but $a_2 \neq 0$, then equation (4) is

$y'(t) = a_2 y(t-\tau)$,which is pure delay equation.

Then characteristic equation becomes

$$me^{m\tau} - a_2 = 0$$
$$\Rightarrow me^{m\tau} = a_2$$
$$\Rightarrow m\tau\, e^{m\tau} = a_2\tau \text{ (Multiplying } \tau \text{ both sides to convert to the inverse of the Lambert function)}$$
$$\Rightarrow Z(m\tau) = a_2\tau$$
$$\Rightarrow m\tau = W(a_2\tau)$$

Under any of these conditions $W(a_2\tau)$ has infinitely many complex roots $\mathrm{r_k} + i\mathrm{s_k}$.

Hence, the solution of equation (4) with $a_1 = 0$ using MOC is

$$\mathrm{y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} + \sum_{k=1}^{\infty} e^{r_k t}\left[c_{(1,k)}\cos(S_k t) + c_{(2,k)}\sin(S_k t)\right]}$$

with $\mathrm{c_1}$ or $\mathrm{c_2}$ zero or not zero according to the restrictions on a_2 and $-\frac{1}{\tau \mathrm{e}}$ and to determine the coefficients $\mathrm{c_1, c_2, c_{(1,k)}, c_{(2,k)}}$, we can use the history function $\phi(\mathrm{t})$ and its derivatives.

(III) In equation (4), when both a_1, a_2 are not zero,

then from characteristic equation (5),

$$(m - a_1)e^{m\tau} - a_2 = 0$$

Let $m - a_1 = n$

$$\Rightarrow ne^{m\tau} - a_2 = 0$$
$$\Rightarrow ne^{(n+a_1)\tau} - a_2 = 0$$
$$\Rightarrow ne^{n\tau} e^{a_1\tau} = a_2$$
$$\Rightarrow n\tau e^{n\tau} = a_2 \tau e^{-a_1\tau}$$
$$\Rightarrow Z(n\tau) = a_2\tau e^{-a_1\tau} \text{ (using Lambert function)}$$
$$\Rightarrow n\tau = W(a_2\tau e^{-a_1\tau})$$

$$\Rightarrow n = \frac{1}{\tau} W(a_2 \tau e^{-a_1 \tau})$$

$$\Rightarrow a_1 + n = a_1 + \frac{1}{\tau} W(a_2 \tau\, e^{-a_1 \tau})$$

$\Rightarrow m = a_1 + \frac{1}{\tau} W(a_2 \tau\, e^{-a_1 \tau})$, which is real and complex satisfying characteristic equation (5).

1.1.2 Steps to solve system of equations (2) and (3) using MOS :

$$y'(t) = a_1(t) y(t) + a_2(t) y(t-\tau),\ t \in [0, \tau]$$
$$y(t) = \phi(t), \quad t \in [-\tau, 0]$$

Step – 1 :

On the interval $[-\tau, 0]$, the function y(t) is the given function $\phi(t)$, so y(t) is known there. So we say the equation is solved for the interval $[-\tau, 0]$ and consider the solution as $y_1(t)$ i.e, $y_1(t) = y_0(t)$.

Note : When $t \in [0, \tau], t - \tau \in [-\tau, 0]$

$\Rightarrow y(t-\tau)$ becomes $y_0(t-\tau)$ on $[0, \tau]$

Step – 2 :

On the interval $[0, \tau]$, the systems (2) and (3) becomes

$$\begin{cases} y'(t) = a_1(t) y(t) + a_2(t) y_0(t-\tau), \text{on } [0, \tau] \\ y(0) = \phi(0) \end{cases} \qquad \ldots(6)$$

The equation (6) is an ODE and not a DDE because $y_0(t-\tau)$ is known and $y_0(t-\tau)$ is simply $\phi(t-\tau)$, thuswe solve this ODE on $[0, \tau]$ using $y(0) = \phi(0)$ as initial condition and the solution on the interval $[0, \tau]$ denote as $y_1(t)$ i.e, $y(t) = y_1(t)$

Note : The equation

$$y'(t) = a_1(t) y(t) + a_2(t) y_0(t-\tau)$$
$$\Rightarrow y'(t) - a_1(t) y(t) = a_2(t) \phi(t-\tau)$$

and $y(0) = \phi(0)$ can be solved using the techniques as we study earlier.

Step 3 :

On the interval $[\tau, 2\tau]$, the system becomes

$$\begin{cases} y'(t) = a_1(t) y(t) + a_2(t) y_1(t-\tau), on\ [\tau, 2\tau] \\ y(\tau) = y_1(\tau) \end{cases} \qquad \ldots(7)$$

Which is again an ODE.

We solve this, using the initial condition at τ and get a solution and denote it by $y_2(t)$, on $[\tau, 2\tau]$ and these steps may be continued for subsequent intervals.

Example 1 :

Find one step of the solution to the system

$$y'(t) = a_1(t) y(t) + a_2(t) y(t-\tau), t \in [0, \tau]$$
$$y(t) = \phi(t), t \in [-\tau, 0]$$

for the following data

$\tau = 4$

$\phi(t) = \tau - t(t + \tau)$

$a_1 = -1$

$a_2 = 0.4$

Solution :
We have the system of first order DDE (2) – (3)

$$y'(t) = a_1 y(t) + a_2 y(t-\tau), \text{for } t \in [0, \tau]$$
$$y(t) = \phi(t), \qquad \text{for } t \in [-\tau, 0]$$

Assigning the given values

$$y'(t) = -y(t) + 0.4y(t-\tau), \text{ for } t \in [0,4]$$
$$y(t) = \tau - t(t+\tau), \text{for } t \in [-4,0]$$

Here $y(t) = \phi(t) = \tau - t(t+\tau) = y_0(t)$

In the interval $[0,4]$,

$$y'(t) = -y(t) + 0.4 y_0(t-\tau)$$
$$\Rightarrow y'(t) = -y(t) + 0.4\phi(t-\tau), t \in [0,4]$$

and $y(0) = \phi(0)$, which is the initial condition and differential equation is the ODE.

Now, $y'(t) = -y(t) + 0.4\phi(t-\tau)$

$$\Rightarrow y'(t) + y(t) = 0.4[\tau - (t-\tau)t]$$
$$\Rightarrow y'(t) + y(t) = 0.4(4 - t^2 + 4t), \text{ which is a linear D.E.}$$

$I.F. = e^{\int dt} = e^t$ and the solution is

$$y(t)e^t = \int 0.4(4 - t^2 + 4t)e^t dt$$
$$\Rightarrow y(t)e^t = 0.4\left[(4 - t^2 + 4t)e^t - \int (4-2t)e^t dt\right]$$
$$\Rightarrow y(t)e^t = 0.4\left[(4 - t^2 + 4t)e^t - \left\{(4-2t)e^t + 2e^t\right\}\right] + c$$
$$\Rightarrow y(t)e^t = 0.4\left[(4 - t^2 + 4t)e^t - (4-2t)e^t - 2e^t\right] + c$$
$$\Rightarrow y(t)e^t = 0.4(4 - t^2 + 4t - 4 + 2t - 2)e^t + c$$
$$\Rightarrow y(t) = 0.4(6t - t^2 - 2) + ce^{-t}$$

We have, $y(0) = \phi(0)$

$$\Rightarrow (0.4)(-2) + c = \tau$$
$$\Rightarrow -0.8 + c = 4$$
$$\Rightarrow c = 4.8$$

The solution is

$$y(t) = 0.4(6t - t^2 - 2) + 4.8e^{-t}, t \in [0,4]$$

Which can be taken as $y_1(t)$ when finding the solution on [4,8].

Idempotent differential equation :
Idempotent Differential equation (IDE) is defined as

$$\left.\begin{aligned} \frac{d}{dt} y(t) &= f(t, y(t), y(u(t))) \\ y(t_0) &= y_0 \end{aligned}\right\} \qquad \ldots(8)$$

Where u(t) is idempotent i.e, u(u(t))=t and t_0 is a fixed point of u.

Inthis paper our main focus is on study of market behavior by modeling in detail (more precisely market equilibrium).

The term market, we mean the conditions under which the producers sell and consumers buy a certain commodity. The term market is used when only one commodity is being bought and sold. The term multimarket is used when more than one commodity is involved. The price, demand and supply of any one good affects the prices etc. of many other goods and vice versa. Knowledge of actual market behavior is limited and so the types of market considered are based on models.

Demand function :
The demand for a particular commodity or goods depends simultaneously on many factors like its price, the prices of other closely related commodities / products, the tested of the consumer, the incomes of the consumer and the distribution of income within the population .

Supply Function :
The Supply Function of individual firms can be defined for
(i) a very short period during which output level can not vary,
(ii) a short run during which output level can be varied but plant size can not,
(iii) a long run in which all inputs are variable.
The supply function of a perfectly competitive firm states the quantity that it will produce, as a function of market price and can be derived from the first order condition for profit maximization.

Market Equilibrium :
The price, demand and supply of any good affect the prices etc. of many other goods and vice versa. Our aim is to discover how the price of a commodity is determined by the behavior of the consumers and of the sellers. Since the knowledge of the actual market behavior is limited, so the types of market to be considered are based on models.
A single price prevails in an ideal competitive market. Both buyers and sellers are price takers, so that their desires are determined by the aggregate demand function and aggregate supply function.
An equilibrium state of the system is defined to be a state in which there is no tendency for the system to change.
Let D(P) is the aggregate demand function.
S(P) is the aggregate supply function , where P is the price.
If demand equals to supply for $P = P_e$
i.e, D(P)=S(P) , for $\mathrm{P} = \mathrm{P_e}$
Then we say market is in equilibrium at $P = P_e$

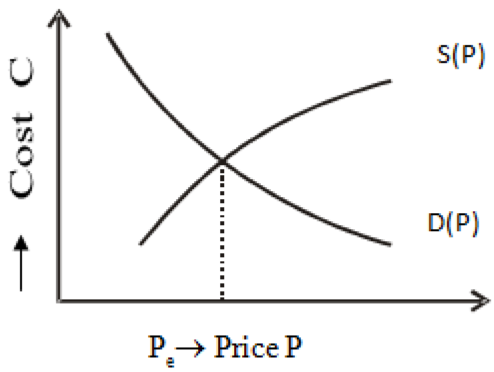

Fig 1 : The usual plot of demand D(P) and supply S(P) giving the equilibrium price P, where they meet.

The market must be dynamic i.e, it should change with time, when not in equilibrium.
If P is a price not satisfying D(P) = S(P), then either
(i) D(P)>S(P) or
(ii)D(P)<S(P)
If D(P)>S(P), then buyers will be willing to pay more than price P for the commodity.
If D(P)<S(P), then supplier incurs unnecessary costs in manufacturing and storing goods that are not sold and will change their supply / production.
In both cases there is a tendency for the market to change.

Existence and Uniqueness of Equilibrium:
Analysis of market equilibrium based upon the assumption that a unique Price-quantity equilibrium exists for each isolated market under investigation.
If unique-price quantity equilibrium does not exist, we mean either supply and demand are not equal at any non-negative price-quantity combination.
Or

Supply and demand are equal at more than one non-negative price-quantity combination.

Special Characteristics of D(P) and S(P) :

A competitive market equilibrium will exist, if there are one or more non-negative prices at which demand and supply are equal and non-negative.

The aggregate demand function D(P) is monotonically decreasing function of P and aggregate supply function is usually monotonically increasing and there is just one value of P where the two curves cross and this is the equilibrium value P_e of the price.

(i) If S(P) >D(P) for all positive P, then in this case, the equilibrium value of P is zero and the commodity is free and consumers can get as much as they want for free good. air is an example of this situation.

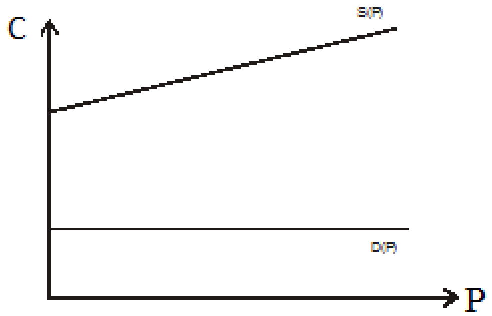

Fig 2 : Supply and demand curve where the equilibrium value is zero.

(ii) If D(P) = S(P) =0

i.e, the price at which a supplier is willing to sell the commodity is greater than any consumer is willing to pay, then supply is non-zero at prices at which the demand is zero.

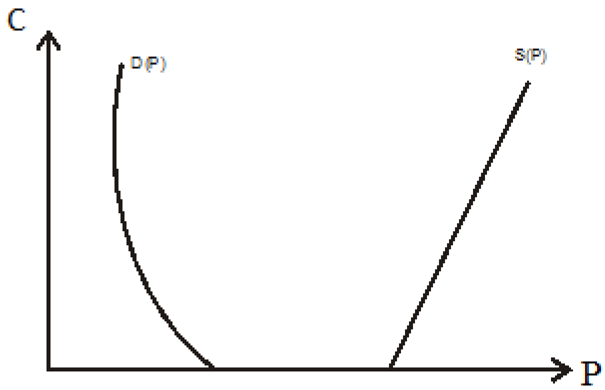

Fig 3: Supply and demand must both be zero for equilibrium here.

(ii) When D(P) > S(P) for every price P, there is no meaningful interpretation that can be placed upon this situation and the models for producer and consumer must altered.

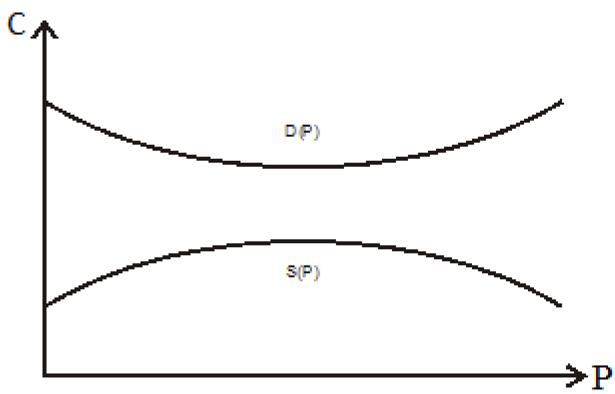

Fig 4 : Demand exceeds supply for every price.

Note : If there is no equilibrium it is probable that model is incorrect.
Uniqueness: Sometimes it is possible that more than one equilibrium exists.
Let $\delta = D'(P) - S'(P)$

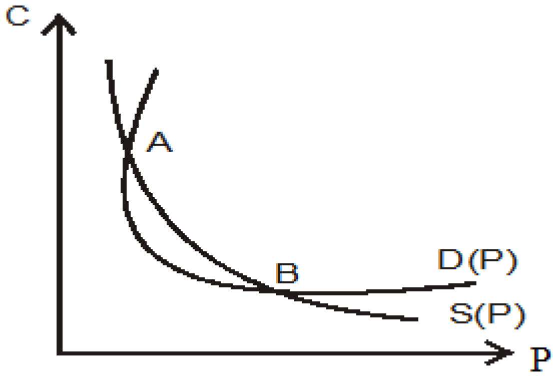

Fig 5 : The aggregate supply S(P) is decreasing and there are two equilibrium values of P.

If the demand curve is negatively sloped throughout and the supply curve is positively sloped throughout, then $\delta < 0$ and there cannot be more than one equilibrium point.

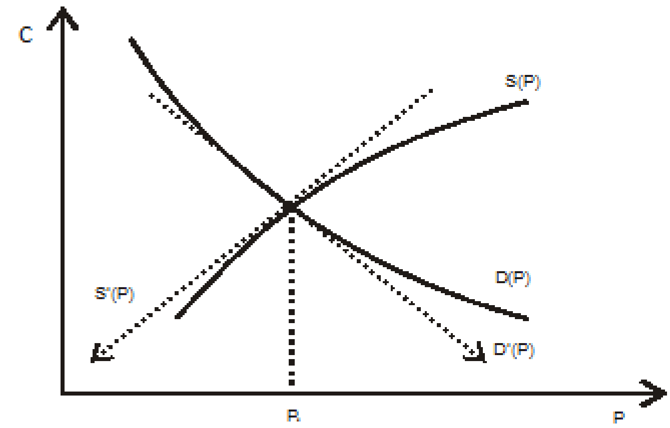

Fig 6 : Unique equilibrium value of P.

If $\delta < 0$ at an equilibrium point P_e, demand will be less than supply at a price slightly higher than P_e and will be greater than supply at a price slightly lower than P_e, hence there can not be a second equilibrium point. The same holds good if $\delta > 0$ through out.

Stability of the equilibrium :
Stability of the equilibrium means when the system is disturbed from equilibrium, will it or will it not return to equilibrium and this depends on the behavior of the market for a single commodity.
The excess demand function is denoted by E(P) and
E(P) = D(P) – S(P)
Where D(P) is the aggregate demand function,
S(P) is the aggregate supply function.
Let P_e be the equilibrium price.
For price P not very different from P_e, using Taylor's theorem,

$$E(P) = E(P_e) + (P - P_e)E'(P_e) + \frac{(P - P_e)^2}{2!}E''(P_e) + \frac{(P - P_e)^3}{3!}E'''(P_e) + \ldots$$

$$\Rightarrow E(P) = E(P_e) + (P - P_e)E'(P_e) + 0(|P - P_e|^2)$$

When $P = P_e, D(P) = S(P)$

So, $\mathrm{E}(\mathrm{P_e}) = 0$

$\Rightarrow \mathrm{E(P)} = (\mathrm{P} - \mathrm{P_e})\mathrm{E}'(\mathrm{P_e}) + 0(|\mathrm{P} - \mathrm{P_e}|^2)$

Let initially the market is disturbed in such a way that E(P) is positive,

then P will increase and E(P) will increase when $\mathrm{E}'(\mathrm{P_e}) > 0$

$\Rightarrow$ The market will return to equilibrium

So, market is instable when $\mathrm{E}'(\mathrm{P_e}) > 0$ and

market is stable when $\mathrm{E}'(\mathrm{P_e}) < 0$

i.e, If a price rise diminishes excess demand and this type of stability is called static stability.

Dynamic Stability :

A positive excess demand tends to raise price can be modeled in many different ways.

A commonly used mathematical model is

$$\left.\begin{array}{l} \dfrac{dP}{dt} = KE(P) \\ P(t_0) = P_0 \end{array}\right\}$$, where K is a +ve constant.

Let $D = aP + b, S = AP + B$

$\Rightarrow D - S = aP + b - AP - B$

$\Rightarrow D - S = (a - A)P + (b - B)$

$\therefore \dfrac{dP}{dt} = K(a - A)P + K(b - B)$, which is a first order linear D.E.

$I.F. = e^{-\int K(a-A)dt} = e^{-K(a-A)t}$ and the solution is

$Pe^{-K(a-A)t} = \int K(b - B)e^{-K(a-A)t} + c$

$$\Rightarrow Pe^{-K(a-A)t} = -K(b - B)\frac{e^{-K(a-A)t}}{K(a - A)} + c$$

$$\Rightarrow P = \frac{-(b - B)}{a - A} + ce^{K(a-A)t} \qquad \ldots(9)$$

When $\mathrm{t} = \mathrm{t_0}, \mathrm{P} = \mathrm{P_0}$

$D - S = (a - A)P + b - B$

When $\mathrm{D} - \mathrm{S} = 0 \Rightarrow \mathrm{P} = \mathrm{P_e}$

$\therefore (a - A)P_e + (b - B) = 0$

$$\Rightarrow P_e = -\left(\frac{b - B}{a - A}\right)$$

So, from equation (9),

$$P_0 = -\frac{b - B}{a - A} + c \qquad (\because \mathrm{P_0} \text{ is initial price at } \mathrm{t} = 0)$$

$\Rightarrow \mathrm{P_0} = \mathrm{P_e} + \mathrm{c}$

$\Rightarrow \mathrm{c} = (\mathrm{P_0} - \mathrm{P_e})$

$\therefore$ The solution is

$P = P_e + (P_0 - P_e)e^{K(a-A)t}$

$\Rightarrow P = (P_0 - P_e)\, e^{K(a-A)t} + P_e$

Thus, equilibrium is stable in the dynamic sense, if the price converges to the equilibrium price over time and unstable if the price change is away from equilibrium.

If P_e is the equilibrium price , then

$P(t) = (P_0 - P_e)e^{K(a-A)t} + P_e$ and

$P(t_0) = P_0, P_0$ is the initial price.

$$\therefore P'(t) = (P_0 - P_e)e^{K(a-A)t}K(a-A)$$
$$= P_0K(a-A)e^{K(a-A)t} - P_eK(a-A)e^{K(a-A)t}$$
$$= KP_0(a-A)e^{K(a-A)t} + KP_e(a-A)(-e^{K(a-A)t})$$

Let $a(t) = e^{K(a-A)t},\ b(t) = -e^{K(a-A)t}$

$$\therefore P'(t) = KP_0(a-A)a(t) + KP_e(a-A)\ b(t)$$
$$\Rightarrow P'(t) = a(t)P(u(t)) + b(t)P(t)$$

Note : When two different systems have the same unique solution, then we will say the two systems are equivalent to each other.

THEOREM : 1

Let u(t) be a function that is its own inverse on an open interval I, $u \in C^{\infty}[I]$ and $t_0 \in I$ be a fixed point of u. Let P_0 be any real number and each of a(t), b(t) be a function of class C^{∞} on I, such that a(t) does not vanish on the interval I , then the system

$$\left.\begin{aligned} &P'(t) = a(t)P(u(t)) + b(t)P(t), on\, I \\ &P(t_0) = P_0 \end{aligned}\right\} \quad \ldots (10)$$

is equivalent to the ordinary second order differential equation system

$$\left.\begin{aligned} &P''(t) = r(t)P'(t) + s(t)P(t)\, on\, I \\ &P(t_0) = P_0 \\ &P'(t_0) = (a(t_0) + b(t_0))P_0 \end{aligned}\right\} \quad \ldots (11)$$

Where $r(t) = \dfrac{a'(t)}{a(t)} + b(t) + b(u(t))u'(t)$

$$s(t) = \frac{-a'(t)b(t)}{a(t)} + b'(t) + a(t)a(u(t))u'(t) - b(t)b(u(t))u'(t)$$

Proof : First to show Equation (10) $\Rightarrow$ Equation (11).

As u(t) is its own inverse on I

$$u^{-1}(t) = u(t)$$
$$\Rightarrow u(u(t)) = t$$

i.e, u(t) is idempotent

To prove (10) $\Rightarrow$ (11).

i.e, To show if P(t) is a solution of the system (10), then it must be a solution of the system (11).we have range of u is I.

So for all $t \in I, u(t)$ is in the domain of a(t) and b(t)

Differentiating equation (10)

$$P''(t) = a(t)P'(u(t))u'(t) + a'(t)P(u(t)) + b(t)P'(t) + b'(t)P(t) \quad \ldots(12)$$

Let u(t)=t, then from equation (10)

$$P'(u(t) = a(u(t))P(u(u(t)) + b(u(t))P(u(t))$$
$$\Rightarrow P'(u(t)) = a(u(t))P(u(u(t))) + b(u(t))P(u(t)) \quad \ldots(13)$$

From equation (10) ,

$$P(u(t)) = \frac{P'(t) - b(t)P(t)}{a(t)} \quad \ldots(14)$$

Using (13) and (14) in equation (12) we get

$$P''(t) = a(t)[a(u(t))P(t) + b(u(t))P(u(t))]u'(t)$$
$$+ a'(t)P(u(t)) + b(t)P'(t) + b'(t)P(t)$$

$$\Rightarrow P''(t) = a(t)\left[a(u(t))P(t) + b(u(t))\left\{ \frac{P'(t) - b(t)P(t)}{a(t)} \right\} \right]u'(t)$$
$$+ a'(t)\left\{ \frac{P'(t) - b(t)P(t)}{a(t)} \right\} + b(t)P'(t) + b'(t)P(t)$$

$$\Rightarrow P''(t) = [a(t)a(u(t))P(t) + b(u(t))P'(t) - b(u(t))b(t)P(t)]u'(t) + \frac{a'(t)}{a(t)}P'(t)$$
$$- \frac{a'(t)}{a(t)}b(t)P(t) + b(t)P'(t) + b'(t)P(t)$$

$$\Rightarrow P''(t) = \left\{ \frac{a'(t)}{a(t)} + b(t) + b(u(t))u'(t) \right\} P'(t)$$
$$+ \left\{ \frac{-a'(t)}{a(t)}b(t) + b'(t) + a(t)a(u(t))u'(t) - b(t)b(u(t))u'(t) \right\} P(t)$$

$$\Rightarrow P''(t) = r(t)P'(t) + s(t)P(t), \text{ on } I$$
$$P(t_0) = P_0$$
$$P'(t_0) = a(t_0)P(u(t_0)) + b(t_0)P(t_0)$$
$$\Rightarrow P'(t_0) = a(t_0)P(t_0) + b(t_0)P(t_0)$$
$$\Rightarrow P'(t_0) = a(t_0)P_0 + b(t_0)P_0$$
$$\Rightarrow P'(t_0) = (a(t_0) + b(t_0))P_0$$

Where

$$r(t) = \frac{a'(t)}{a(t)} + b(t) + b(u(t))u'(t)$$
$$s(t) = -\frac{a'(t)}{a(t)}b(t) + b'(t) + a(t)a(u(t))u'(t) - b(t)b(u(t))u'(t)$$

So, Equation (10) $\Rightarrow$ Equation (11).

Again to show Equation (11) $\Rightarrow$ Equation (10),

Let P(t) be the solution to equation (11), we have to show P(t) satisfies equation (10).

Here P(t) exists and is unique as the system (11) is a second order linearly ordinary D.E. whose coefficients are in $C^\infty[I]$.

Let $L(t) = P'(t) - a(t)P(u(t)) - b(t)P(t)$ …(15)

Where $a(t), b(t) \in C^\infty[I]$

Here $L(t_0) = P'(t_0) - a(t_0)P(u(t_0)) - b(t_0)P(t_0)$

$$= (a(t_0) + b(t_0))P_0 - a(t_0)P(u(t_0)) - b(t_0)P(t_0)$$
$$= a(t_0)P_0 + b(t_0)P_0 - a(t_0)P(u(t_0)) - b(t_0)P_0$$
$$= a(t_0)P_0 + b(t_0)P_0 - a(t_0)P(t_0) - b(t_0)P_0 \quad (\because u(t) = t)$$
$$= a(t_0)P_0 + b(t_0)P_0 - a(t_0)P_0 - b(t_0)P_0$$
$$= 0$$

Again, $L'(t) = P''(t) - a(t)P'(u(t))u'(t) - a'(t)P(u(t)) - b(t)P'(t) - b'(t)P(t)$ … (16)

Also, $L(u(t)) = P'(u(t)) - a(u(t))P(u(u(t))) - b(u(t))P(u(t))$

$\Rightarrow L(u(t)) = P'(u(t)) - a(u(t))P(u(t)) - b(u(t))P(u(t)) \quad (\because u(t) = t)$

Putting the value of $P''(t)$ from equation (11) in equation (16), we get

$$L'(t) = r(t)P'(t) + s(t)P(t) - a(t)P'(u(t))u'(t) - a'(t)P(u(t)) - b(t)P'(t) - b'(t)P(t)$$

$$\Rightarrow L'(t) = \left[\frac{a'(t)}{a(t)} + b(t) + b(u(t))u'(t)\right]P'(t)$$
$$+\left[\frac{-a'(t)b(t)}{a(t)} + b'(t) + a(t)a(u(t))u'(t) - b(t)b(u(t))u'(t)\right]P(t)$$
$$- a(t)P'(u(t))u'(t) - a'(t)P(u(t)) - b(t)P'(t) - b'(t)P(t)$$

$$= \frac{a'(t)}{a(t)}P'(t) + b(t)P'(t) + b(u(t))u'(t)P'(t) - \frac{a'(t)}{a(t)}b(t)P(t)$$
$$+ b'(t)P(t) + a(t)a(u(t))u'(t)P(t) - b(t)b(u(t))u'(t)P(t)$$
$$- a(t)P'(u(t))u'(t) - a'(t)p(u(t)) - b(t)P'(t) - b'(t)P(t)$$
$$= -a(t)P'(u(t))u'(t) + a(t)a(u(t))u'(t)P(t)$$
$$+ a(t)u'(t)b(u(t))P(u(t)) + \frac{a'(t)}{a(t)}p'(t) - \frac{a'(t)}{a(t)}b(t)P(t) - a'(t)p(u(t))$$
$$+ b(u(t))u'(t)P'(t) - a(t)u'(t)b(u(t))P(u(t)) - b(t)b(u(t))u'(t)P(t)$$
$$= -a(t)u'(t)[P'(u(t)) - a(u(t))P(u(t)) - b(u(t))P(u(t))]$$
$$+ \frac{a'(t)}{a(t)}\left[P'(t) - a(t)P(u(t)) - b(t)P(t)\right]$$
$$+ b(u(t))u'(t)\left[P'(t) - a(t)P(u(t)) - b(t)P(t)\right] \quad (\because u(t) = t)$$
$$= -a(t)u'(t)L(u(t)) + \left\{\frac{a'(t)}{a(t)} + b(u(t))u'(t)\right\}\left\{P'(t) - a(t)P(u(t)) - b(t)P(t)\right\}$$
$$= -a(t)u'(t)L(u(t)) + \left\{\frac{a'(t)}{a(t)} + b(u(t))u'(t)\right\}L(t)$$

$\Rightarrow \mathrm{L}'(\mathrm{t}) = \mathrm{A}(\mathrm{t})\mathrm{L}(\mathrm{u}(\mathrm{t})) + \mathrm{B}(\mathrm{t})\mathrm{L}(\mathrm{t})$ where

$$A(t) = -a(t)u'(t), B(t) = \frac{a'(t)}{a(t)} + b(u(t))u'(t)$$

Thus, we get

$$\left.\begin{aligned} &\mathrm{L}'(\mathrm{t}) = \mathrm{A}(\mathrm{t})\mathrm{L}(\mathrm{u}(\mathrm{t})) + \mathrm{B}(\mathrm{t})\mathrm{L}(\mathrm{t}) \\ &\mathrm{L}(\mathrm{t}_0) = 0 \end{aligned}\right\} \quad \ldots (17)$$

The coefficients A(t) and B(t) and the initial condition in (11) are sufficient to establish that,

$$\mathrm{L}(\mathrm{t}) = 0$$
$$\Rightarrow y'(t) - a(t)P(u(t)) - b(t)P(t) = 0$$
$$\Rightarrow y'(t) = a(t)P(u(t)) + b(t)P(t),$$

and $P(t_0) = P_0$

This shows that P(t) in equation (11) is the solution to equation (10).
So,Equation (11) $\Rightarrow$ Equation (10)
Hence the systems are equivalent to each other.

Example -2 :

In equation (10), let I be the interval (1,5),

$a(t) = e^{K(a-A)t}$, where $K(a-A) = 0.1347,$

$b(t) = 0.5, P_0 = 1, u(t) = \frac{5}{t}$.

Fixed point of u is $t_0 = \sqrt{5}$.

Solution :

The equation (10) looks like

$$P'(t) = a(t)P(u(t)) + b(t)P(t)$$
$$P(t_0) = P_0$$
$$\Rightarrow P'(t) = e^{K(a-A)t} P\left(\frac{5}{t}\right) + 0.5P(t)$$
$$p\left(\sqrt{5}\right) = 1$$

$$\left.\begin{aligned} &\Rightarrow P'(t) = e^{0.1347t} P\left(\frac{5}{t}\right) + 0.5P(t), t \in (1,5) \\ &P\left(\sqrt{5}\right) = 1 \end{aligned}\right\} \quad \dots (18)$$

and this is equivalent to the ordinary second order D.E.

$$P''(t) = r(t)P'(t) + s(t)P(t),\ t \in (1,5)$$
$$P(t_0) = P_0$$
$$P'(t_0) = (a(t_0) + b(t_0))P_0$$
$$\Rightarrow P''(t) = r(t)P'(t) + s(t)P(t),\ t \in (1,5)$$
$$P\left(\sqrt{5}\right) = 1$$
$$P'\left(\sqrt{5}\right) = \left(e^{0.1347\times\sqrt{5}} + 0.5\right)1 = e^{0.3011} + 0.5 = 1.8513$$

where $r(t) = \frac{a'(t)}{a(t)} + b(t) + b(u(t))\ u'(t)$

$$= \frac{0.1347e^{0.1347t}}{e^{0.1347t}} + 0.5 + 0.5\left(\frac{-5}{t^2}\right)$$
$$= 0.1347 + 0.5 - \frac{2.5}{t^2}$$
$$= 0.6347 - \frac{2.5}{t^2} = \frac{0.6347t^2 - 2.5}{t^2}$$
$$s(t) = \frac{-a'(t)b(t)}{a(t)} + b'(t) + a(t)a(u(t))u'(t) - b(t)b(u(t))u'(t)$$
$$= \frac{-0.1347e^{0.1347t} \times 0.5}{e^{0.1347t}} + 0 + e^{0.1347t} e^{(0.1347)\frac{5}{t}} - 0.5 \times 0.5\left(-\frac{5}{t^2}\right)$$
$$= -0.06735 + e^{0.1347t + \frac{5(0.1347)}{t}} + \frac{12.5}{t^2}$$
$$= \frac{12.5}{t^2} + e^{\frac{0.1347t^2 + 0.6735}{t}} - 0.06735$$
$$= \frac{12.5 + t^2 e^{\frac{0.1347t^2 + 0.6735}{t}} - 0.06735}{t^2}$$

$$\left.\begin{aligned} &so,\ P''(t)=r(t)P'(t)+s(t)P(t), t\in(1,5)\\ &r(t)=\frac{0.6347t^2-2.5}{t^2}\\ &s(t)=\frac{12.5+t^2e^{\frac{0.1347t^2+0.6735}{t}}-0.06735t^2}{t^2}\\ &p\left(\sqrt{5}\right)=1\\ &P'\left(\sqrt{5}\right)=1.8513 \end{aligned}\right\} \quad \ldots (19)$$

Now, $P'(t)=e^{0.1347t}P\left(\frac{5}{t}\right)+0.5P(t)$

$$\Rightarrow \frac{dP}{dt}-0.5P(t)=e^{0.1347t}P\left(\frac{5}{t}\right)$$

Let $P(t)=v$

$$\Rightarrow \frac{dv}{dt}-0.5v=e^{0.1347t}P\left(\frac{5}{t}\right)$$

Let $P\left(\frac{5}{t}\right)=1$

$\Rightarrow \frac{\mathrm{dv}}{\mathrm{dt}}-0.5\mathrm{v}=\mathrm{e}^{0.1347t}$, which is a linear D.E.

Integrating factor (I.F) $=\mathrm{e}^{-0.5t}$

So, $ve^{-0.5t}=\int e^{-0.5t}e^{0.1347t}dt$

$$\Rightarrow ve^{-0.5t}=\int e^{(0.1347-o.5)t}dt$$

$$\Rightarrow \mathrm{ve}^{-0.5t}=\int \mathrm{e}^{-0.3653t}\mathrm{dt}$$

$$\Rightarrow \mathrm{ve}^{-0.5t}=\frac{\mathrm{e}^{-0.3653t}}{-0.3653}+\mathrm{c}$$

$$\Rightarrow \mathrm{v}=\frac{\mathrm{e}^{(-0.3653+0.5)t}}{-0.3653}+\mathrm{ce}^{0.5t}$$

$$\Rightarrow \mathrm{v}=\frac{\mathrm{e}^{0.1347t}}{-0.3653}+\mathrm{ce}^{0.5t}$$

$$\Rightarrow P(t)=\frac{e^{0.1347t}}{-0.3653}+ce^{0.5t}$$

As $P\left(\sqrt{5}\right)=1$

$$\Rightarrow 1=\frac{\mathrm{e}^{0.1347\times\sqrt{5}}}{-0.3653}+\mathrm{ce}^{0.5\times\sqrt{5}}$$

$$\Rightarrow 1=\frac{1.3514}{0.3653}+\mathrm{c}(3.0588)$$

$$\Rightarrow 1=-3.6994+\mathrm{c}(3.0588)$$

$$\Rightarrow (3.0588)c=1+3.6994$$

$\Rightarrow \mathrm{c}=\frac{4.6994}{3.0588}=1.5363$ (Approx.)

$$\therefore P(t) = \frac{e^{0.1347t}}{-0.3653} + 1.5363 \times e^{0.5t}$$

$$\Rightarrow P'(t) = \frac{0.1347e^{0.1347t}}{-0.3653} + (0.5)(1.5363)e^{0.5t}$$

$$P''(t) = \frac{(0.1347)^2 e^{0.1347t}}{-0.3653} + (0.5)(0.5)(1.5363) \times e^{0.5t}$$

$$\Rightarrow P''(t) = (-0.0496)\, e^{(0.1347)t} + (0.3840)e^{0.5t} \qquad \ldots (20)$$

Again $r(t)\, P'(t) + s(t)\ p(t)$

$$= \left(\frac{0.6347\mathrm{t}^2 - 2.5}{\mathrm{t}^2}\right)\left(\frac{0.1347\mathrm{e}^{0.1347\mathrm{t}}}{-0.3653} + 0.5 \times 1.5363 \times \mathrm{e}^{0.5\mathrm{t}}\right)$$

$$+ \left(\frac{12.5 + t^2 e^{\frac{0.1347t^2+0.6735}{t}} - 0.06735t^2}{t^2}\right)\left(\frac{e^{0.1347t}}{-0.3653} + 1.5363 \times e^{0.5t}\right) \qquad \ldots (21)$$

As, $\mathrm{b(t)} = 0.5$

$\Rightarrow \mathrm{e}^{-\mathrm{K(a-A)t}} = 0.5$

$\Rightarrow \mathrm{e}^{-0.1347t} = 0.5$

$\Rightarrow -0.1347\mathrm{t} = \log(0.5)$

$$\Rightarrow \mathrm{t} = \frac{\log(0.5)}{-0.1347}$$

$\Rightarrow \mathrm{t} = 2.23 \in (1,5)$, taking logarithmic with base 10.

Using t = 2.23, we conclude that the equations (18) and (19) are equivalent.

II. Conclusion

In this paper, we have introduced Delay Differential Equations involving market equilibrium. Further two methods MOC and MOS are used in solving DDEs. We have also shown that differentiation can be used as a method for solving IDEs. Also stability of the equilibrium is discussed in the theorem of this paper. This theorem can also be extended to the ordinary third order differential equation.

References

[1] ASl, F.M. and Ulsoy, A.G., Analysis of a system of Linear Delay Differential Equations, Journal of Dynamic Systems, Measurement and Control,2003, V.125, PP215-223.

[2] Bender, A.E., and Neuwirth, L.P. Traffic Flow : Laplace Transforms, American Mathematical Monthly, Vol.80, (1973), 417-423.

[3] Driver, R.D., Sasser, D.W and Slater, M.L., The Equation y'(t)=ay(t)+by(t-τ) with 'small' Delay, American Mathematical Monthly, Vol.80, 1973, pp 990-995.

[4] Falbo, C.E. Complete Solutions to Certain Functional Differential Equations, Proceedings systemic, Cybernetics, informatics, 1998, Vol.4, pp.157-161.

[5] Flabo, C.E., Idempotent Differential Equation, Journal of interdisciplinary Mathematics, Vol.6, No.3, pp279-289, October 2003.

[6] Heffernam, J.M. and Corless, R.M., Solving some Delay Differential Equations with computer algebra, Applied probability Trust, 2005, pp 1-22

[7] Myshkis, A.D., Differential Equations, Ordinary with Distributed Arguments, Encyclopedia of Mathematics, Vol.3, Kluwer Academic Publishers, Boston, 1989, pp 144-147

[8] Nesbit, R., Delay Differential Equations for structured populations, Structured population models in marine , Terrestrial and Freshwater systems, 1997.

[9] Ryder, G.H., Solutions of a Functional Differential Equation, American Mathematical Monthly, Vol. 76(1969), 1031-1033.