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Bound Estimation of Two Functions with Proofs of Some New Inequalities

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Abstract : *This article presents a bound estimation for two functions and consequently puts forwards several inequalities with their proofs. The new inequalities are helpful in estimating the bounds of certain functions and their better proofs are still calling more mathematical skills.*

Keywords: *Analytic solution, bound estimation, inequality*

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I. Introduction

A recent research came across the problem of estimating the bounds of function $f(x)=\frac{1}{\sqrt{x}}-\frac{2}{\sqrt{x+1}}+\frac{1}{\sqrt{x+2}}$ and function $f(x)=\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{x+1}}+\frac{2}{\sqrt{x+2}}$ in analytic expressions. Compared with the ordinary problem of finding a function's extremum, this problem is indeed a time-consuming one for its requests of 'analytic expressions', which means that one ought to deduce and derive out the solution in analytic expressions. A technical engineer will naturally look into the classical handbooks of inequalities for some guidance. But unfortunately, to this problem, he/she will be disappointed because there is little information in present literatures, such as the bibliographies [1] to [8]. Hence finding a solution is mandatory.

This article presents solution to the problem. Through mathematical deductions and proofs, the article derives out for each function the analytic solutions, which form several inequalities. The analytic solution can be utilized in engineering modeling.

II. Main Results and Proofs

Theorem 1. Let $f(x)=\frac{1}{\sqrt{x}}-\frac{2}{\sqrt{x+1}}+\frac{1}{\sqrt{x+2}}$ with $x>0$; then $f(x)>0$ and $f'(x)<0$.

Proof. Let $l(x)=\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+1}}$ and $r(x)=\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x+2}}$; then it yields

$$l(x)=\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+1}}=\frac{1}{(\sqrt{x+1}+\sqrt{x})\sqrt{x(x+1)}} \tag{1}$$

and

$$r(x)=\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x+2}}=\frac{1}{(\sqrt{x+1}+\sqrt{x+2})\sqrt{(x+1)(x+2)}} \tag{2}$$

When $x>0$ it obviously holds

$$l(x)>r(x)$$

and thus $f(x)=l(x)-r(x)>0$.

Direct calculation yields

$$\begin{aligned} f'(x) &= -\frac{1}{2x\sqrt{x}} + \frac{1}{(x+1)\sqrt{x+1}} - \frac{1}{2(x+2)\sqrt{x+2}} \\ &= -\frac{1}{2}\left(\frac{1}{\sqrt{x^3}} - \frac{2}{\sqrt{(x+1)^3}} + \frac{1}{\sqrt{(x+2)^3}}\right) \\ &= -\frac{1}{2}\left(\frac{1}{\sqrt{x^3}} - \frac{2}{\sqrt{(x+1)^3}} + \frac{1}{\sqrt{(x+2)^3}}\right) \end{aligned} \tag{3}$$

Since $x(x+2) < (x+1)^2$, it holds $x^3(x+2)^3 < (x+1)^6$ and thus

$$\frac{1}{\sqrt{x^3}} + \frac{1}{\sqrt{(x+2)^3}} \geq \frac{2}{\sqrt[4]{x^3(x+2)^3}} > \frac{2}{\sqrt[4]{(x+1)^6}} = \frac{2}{\sqrt{(x+1)^3}} \tag{4}$$

Obviously, (3) and (4) result in $f'(x) < 0$.□

Corollary 1. Let $f(x) = \frac{1}{\sqrt{x^\alpha}} - \frac{2}{\sqrt{(x+1)^\alpha}} + \frac{1}{\sqrt{(x+2)^\alpha}}$ with $x > 0$ and $\alpha > 0$; then $f(x) > 0$ and $f'(x) < 0$.

Proof. $x(x+2) < (x+1)^2$ yields $x^\alpha(x+2)^\alpha < (x+1)^{2\alpha}$ and thus

$$\frac{1}{\sqrt{x^\alpha}} + \frac{1}{\sqrt{(x+2)^\alpha}} \geq \frac{2}{\sqrt{\sqrt{x^\alpha}\sqrt{(x+2)^\alpha}}} > \frac{2}{\sqrt{(x+1)^\alpha}} \tag{5}$$

which is sure that $f(x) > 0$.

Direct calculation shows

$$f'(x) = -\frac{\alpha}{2}\left(\frac{1}{\sqrt{x^{\alpha+2}}} - \frac{2}{\sqrt{(x+1)^{\alpha+2}}} + \frac{1}{\sqrt{(x+2)^{\alpha+2}}}\right) \tag{6}$$

Referring the proof of (5), it knows $f'(x) < 0$.□

Theorem 2. Let $f(x) = \frac{1}{\sqrt{x}} - \frac{2}{\sqrt{x+1}} + \frac{1}{\sqrt{x+2}}$ and $g(x) = \frac{1}{x(x+1)(x+2)}$ with $x \geq 2$; then $f(x) > g(x)$.

Proof. Let $\omega(x) = \frac{1}{\sqrt{x}} - \frac{1}{2x}$, where $x \geq 2$; then it holds

$$\omega(x) = \frac{2\sqrt{x}-1}{2x} > 0$$

$$\omega'(x) = -\frac{\sqrt{x}-1}{2x^2} < 0$$

$$\omega''(x) = \frac{(3\sqrt{x}-4)}{4x^2} > 0$$

Hence $\omega(x)$ is a strict concave upward, and it fits for arbitrary $x_0 \in [2,\infty)$

$$\omega(\frac{x_0 + x_0 + 2}{2}) < \frac{1}{2}(\omega(x_0) + \omega(x_0 + 2))$$

Namely

$$\omega(x_0) + \omega(x_0 + 2) > 2\omega(x_0 + 1)$$

That is

$$\frac{1}{\sqrt{x_0}} - \frac{1}{2x_0} + \frac{1}{\sqrt{x_0 + 2}} - \frac{1}{2(x_0 - 2)} > 2(\frac{1}{\sqrt{x_0 + 1}} - \frac{1}{2(x_0 + 1)})$$

which yields

$$\frac{1}{\sqrt{x_0}} - \frac{2}{\sqrt{x_0 + 1}} + \frac{1}{\sqrt{x_0 + 2}} > \frac{1}{2x_0} - \frac{1}{x_0 + 1} + \frac{1}{2(x_0 - 2)}$$

Since $\frac{(x_2 + 1)(x_0 + 2) - 2x_0(x_0 + 2) + x_0(x_0 + 1)}{2x_0(x_0 + 1)(x_0 + 2)} = \frac{1}{x_0(x_0 + 1)(x_0 + 2)}$, it knows

$$\frac{1}{\sqrt{x_0}} - \frac{2}{\sqrt{x_0 + 1}} + \frac{1}{\sqrt{x_0 + 2}} > \frac{1}{x_0(x_0 + 1)(x_0 + 2)} \tag{7}$$

Owning to the arbitrariness of x_0, the Theorem 2 holds.

Corollary 2. Let x and α be real numbers with $x \ge 2$ and $\alpha > 0$; then

$$\frac{1}{\sqrt{x^\alpha}} - \frac{2}{\sqrt{(x+1)^\alpha}} + \frac{1}{\sqrt{(x+2)^\alpha}} > \frac{1}{x^\alpha (x+1)^\alpha (x+2)^\alpha} \tag{8}$$

Proof. (Omitted) □

Corollary 3. Let x and α be real numbers with $x \ge 2$ and $\alpha > 0$; then

$$\frac{1}{\sqrt{x^\alpha}} - \frac{2}{\sqrt{(x+1)^\alpha}} + \frac{1}{\sqrt{(x+2)^\alpha}} > \frac{1}{(x+1)^{3\alpha}} > \frac{1}{(x+2)^{3\alpha}} \tag{9}$$

Proof. $x(x+2) < (x+1)^2 \Rightarrow \frac{1}{x(x+2)} > \frac{1}{(x+1)^2}$; then (8) directly derives out (9). □

Theorem 3. Let $f(x) = \frac{1}{\sqrt{x}} - \frac{3}{\sqrt{x+1}} + \frac{2}{\sqrt{x+2}}$ with $x \ge \frac{2(\sqrt[3]{4} + 2\sqrt[3]{2} + 1)}{3} (\approx 3.4)$; then $f(x) \le 0$ and $f'(x) \ge 0$

Proof. First let $l(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$ and $r(x) = \frac{2}{\sqrt{x+1}} - \frac{2}{\sqrt{x+2}}$ with $x > 0$; then it holds

$$l(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}} = \frac{1}{x(x+1)(\sqrt{x} + \sqrt{x+1})} \tag{10}$$

$$r(x)=\frac{2}{\sqrt{x+1}}-\frac{2}{\sqrt{x+2}}=\frac{2}{(x+1)(x+2)(\sqrt{x+1}+\sqrt{x+2})}=\frac{1}{(x+1)(\frac{x+2}{2})(\sqrt{x+1}+\sqrt{x+2})} \tag{11}$$

Note that, when $x\ge\frac{2(\sqrt[3]{4}+2\sqrt[3]{2}+1)}{3}$, $\frac{x+2}{2}\le x$ and $(\frac{x+2}{2})\sqrt{x+2}\le x\sqrt{x}$; comparing (11) with (10) it knows $r(x)>l(x)$, namely,

$$\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+1}}\le\frac{2}{\sqrt{x+1}}-\frac{2}{\sqrt{x+2}}$$

which is

$$\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{x+1}}+\frac{2}{\sqrt{x+2}}\le 0 \tag{12}$$

Note that

$$f'(x)=-\frac{1}{2}(\frac{1}{x^{\frac{3}{2}}}-\frac{3}{(x+1)^{\frac{3}{2}}}+\frac{2}{(x+2)^{\frac{3}{2}}})$$

it knows $f'(x)\ge 0$ by letting $y=x^3$ and referring to (12). □

Theorem 4. Let $f(x)=\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{x+1}}+\frac{2}{\sqrt{x+2}}$ and with $x>(\frac{17}{6})^2(\approx 8.02)$; then

$$-\frac{2}{3\sqrt{x(x+1)(x+2)}}<f(x)<-\frac{1}{3\sqrt{x(x+1)(x+2)}} \tag{13}$$

Proof. Direct calculation shows

$$\begin{aligned}f(x)&=\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{x+1}}+\frac{2}{\sqrt{x+2}}+\frac{1}{3\sqrt{x(x+1)(x+2)}}\\&=\frac{3\sqrt{(x+1)(x+2)}-9\sqrt{x(x+2)}+6\sqrt{x(x+1)}+1}{3\sqrt{x(x+1)(x+2)}}\end{aligned}$$

and

$$\begin{aligned}f(x)&=\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{x+1}}+\frac{2}{\sqrt{x+2}}+\frac{2}{\sqrt{x(x+1)(x+2)}}\\&=\frac{\sqrt{(x+1)(x+2)}-3\sqrt{x(x+2)}+2\sqrt{x(x+1)}+2}{\sqrt{x(x+1)(x+2)}}\end{aligned}$$

Let

$$h(x)=3\sqrt{(x+1)(x+2)}-9\sqrt{x(x+2)}+6\sqrt{x(x+1)}+1 \tag{14}$$

and

$$e(x)=3\sqrt{(x+1)(x+2)}-9\sqrt{x(x+2)}+6\sqrt{x(x+1)}+2 \tag{15}$$

Since $x+\frac{23}{16}<\sqrt{(x+1)(x+2)}<x+\frac{3}{2}$ with $x>2\times(\frac{23}{8})^2-16$, $x+\frac{17}{18}<\sqrt{x(x+2)}<x+1$ with $x>(\frac{17}{6})^2$ and $x+\frac{15}{32}<\sqrt{x(x+1)}<x+\frac{1}{2}$ with $x>(\frac{15}{8})^2$, it yields when $x>(\frac{17}{6})^2$

$$h(x)<3(x+\frac{3}{2})-9(x+\frac{17}{18})+6(x+\frac{1}{2})+1=0$$

and

$$e(x)>3(x+\frac{23}{16})-9x-9+6(x+\frac{15}{32})+2=\frac{1}{8}>0$$

Hence when $x>(\frac{17}{6})^2$, inequality (13) holds. □

Corollary 4 Let I be positive integer; then

$$|\frac{1}{\sqrt{i}}-\frac{1}{\sqrt{i+1}}|>|\frac{1}{\sqrt{i+1}}-\frac{1}{\sqrt{i+2}}| \tag{16}$$

$$\frac{1}{\sqrt{i}}-\frac{2}{\sqrt{i+1}}+\frac{1}{\sqrt{i+2}}>\frac{1}{i(i+1)(i+2)}-\frac{1}{6}, i=1 \tag{17}$$

$$\frac{1}{\sqrt{i}}-\frac{2}{\sqrt{i+1}}+\frac{1}{\sqrt{i+2}}>\frac{1}{i(i+1)(i+2)}, i>1 \tag{18}$$

$$-\frac{2}{3\sqrt{i(i+1)(i+2)}}<\frac{1}{\sqrt{i}}-\frac{3}{\sqrt{i+1}}+\frac{2}{\sqrt{i+2}}<-\frac{1}{3\sqrt{i(i+1)(i+2)}}, i>8 \tag{19}$$

Proof. Only for (17) because (18) is the integer form of Theorem 2 and (19) is the integer form that comes from Theorem 3 and Theorem 4. When $i=1$, the right side is $\frac{1}{1\times2\times3}-\frac{1}{6}=0$ while the left side is, by Theorem 1, a positive number. □

III. Discussions and Expectations

By Theorem 2 and Theorem 4, one can summarizes that

$$\frac{1}{\sqrt{x}}-\frac{2}{\sqrt{x+1}}+\frac{1}{\sqrt{x+2}}>\frac{1}{x(x+1)(x+2)}, x\in[2,\infty) \tag{20}$$

and

$$-\frac{2}{3\sqrt{x(x+1)(x+2)}}<\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{x+1}}+\frac{2}{\sqrt{x+2}}<-\frac{1}{3\sqrt{x(x+1)(x+2)}}, x\in((\frac{17}{6})^2,\infty) \tag{21}$$

However, it is necessary to point out that, these two inequalities are both very rough though they can meet with needs of some engineering evaluation. According to Mathematica's experiments, as shown in figure 1, the inequality (20) should be hold on interval $[\alpha,\infty)$, where $\alpha\approx1.0636$. This is more accurate than (20).

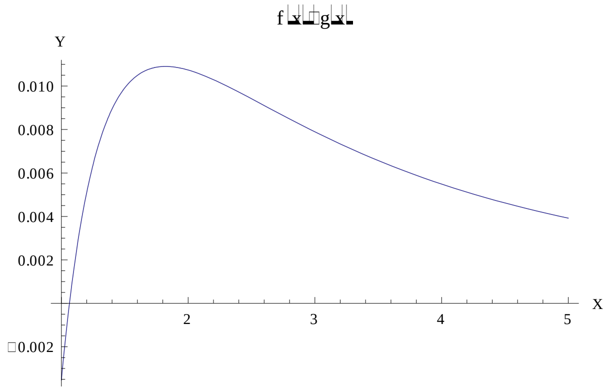

Fig.1 A Zero point in [1,2]

Meanwhile, Corollary 3 says

$$\frac{1}{\sqrt{x^{\alpha}}}-\frac{2}{\sqrt{(x+1)^{\alpha}}}+\frac{1}{\sqrt{(x+2)^{\alpha}}}>\frac{1}{(x+1)^{3\alpha}}>\frac{1}{(x+2)^{3\alpha}},x\in[2,\infty) \qquad (22)$$

but Mathematica's experiments show that there is a C and β with $1\le C<4$ and $\beta>2$, satisfying

$$\frac{1}{\sqrt{x}}-\frac{2}{\sqrt{x+1}}+\frac{1}{\sqrt{x+2}}>\frac{C}{x^{3}}>\frac{C}{(x+1)^{3}}>\frac{C}{(x+2)^{3}},x\in[\beta,\infty)$$

What is the range of C and β ? This article cannot answer the question.

In (21), the range of x is limited to $x\in((\frac{17}{6})^{2},\infty)$, which is the best result this article can obtain, but Mathematica's experiments show that, the range of x should be limited to (α,∞) with $\alpha\approx 5.728$, as shown in figure 2.

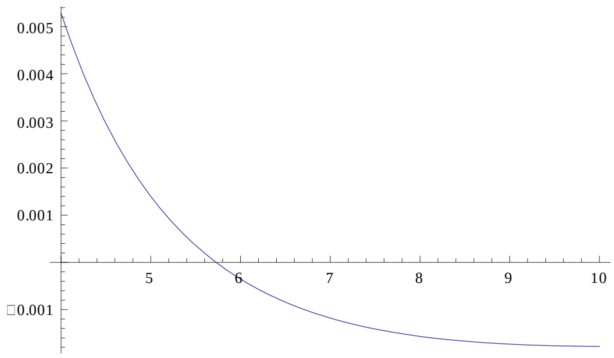

FIG.2 A ZERO POINT AROUND 5.728

In addition, Mathematica's experiments also show that, there are real number A, B and γ such that

$$-\frac{A}{\sqrt{x(x+1)(x+2)}}<\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{x+1}}+\frac{2}{\sqrt{x+2}}<-\frac{B}{\sqrt{x(x+1)(x+2)}},x\in[\gamma,\infty) \qquad (23)$$

For example, when $A=2$, $B=-2.5$ it holds

$$-\frac{2}{\sqrt{x(x+1)(x+2)}}<\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{x+1}}+\frac{2}{\sqrt{x+2}}<0<\frac{2.5}{\sqrt{x(x+1)(x+2)}},x\in(0,\infty) \qquad (24)$$

How to determine the ranges of A, B and γ remains a problem that cannot be solved in this article. In the end, Mathematica also shows

$$-\frac{512}{(x+2)\sqrt{x+2}} < \frac{1}{\sqrt{x}} - \frac{3}{\sqrt{x+1}} + \frac{2}{\sqrt{x+2}} < -\frac{1}{512x^3}, x \in (1.31023, \infty) \tag{25}$$

which this article either remains not proving.

All these point to future studies. All in all, the results presented in this article need refining and hope to see better results in the future.

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