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On the Estimation of the Bivariate Exponentiated Pareto Distribution

Mervat k. Abd Elaal [1, 2], Ashwag S. ALerwi[1]
[1] Statistics Department, Faculty of Sciences King Abdulaziz University Jeddah, Kingdom of Saudi Arabia
[2] Statistics Department, Faculty of Commerce Al-Azhar University, Girls Branch Cairo, Egypt

Abstract: *In this paper, a new bivariate exponentiated Pareto distribution is introduced. The proposed bivariate distribution is constructed based on Gaussian copula with exponentiated Pareto distribution as marginals. Several properties of the proposed bivariate distribution can be obtained using the Gaussian copula property. Moreover, different methods of estimation are considered to estimate the unknown parameters of proposed bivariate distribution and their performances are compered through numerical simulations.*

Key words: *exponentiated Pareto distribution; copula; Gaussian copula; M mixture representation; maximum likelihood estimators; Bayesian estimator*

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I. Introduction

The Pareto distribution has many important applications. It is used to describe geophysical, social, scientific, actuarial, and other types of phenomena. However, the Pareto distribution does not provide a reasonable parametric fit for some practical applications where the underlying hazard rates are non decreasing. Gupta et al. (1998) introduced a new two-parameter distribution as a generalization of the standard Pareto of the second kind, called the exponentiated Pareto distribution. They proved that the exponentiated Pareto distribution is effective in analyzing many lifetime data. The failure rates take decreasing and upside-down bathtub shapes depending on the value of the shape parameter.

In this situation, we could consider some existing bivariate lifetime distribution that has been introduced in the literature such as the studies of Freund (1961), Marshall and Olkin (1967), Mardia (1970), Sarhan and Balakrishnan (2007), Kundu et al. (2010), Gupta et al. (2010), Al-Mutairi et al. (2011), Sankaran et al. (2014), and Olkin and Trikalinos (2015). A flexible way to derive different bivariate lifetime distributions could be given by copula function, see for example, Adham and Walker (2001), AL-Hussaini and Ateya (2006), Al-Dayian et al. (2008), Quiroz-Flores (2009),Gupta et al. (2010), Kundu and Gupta (2011), El-Sherpieny et al. (2013), Kundu (2014), Achcar et al. (2015) and El-Gohary and El-Morshedy (2015).

The main aim of this paper is to introduce a bivariate exponentiated Pareto distribution with a bivariate gamma mixing distribution based on Gaussian copula. In addition, estimate of the parameters will be investigated and analyzed. We will also use simulation study to examine the performance of this new distribution and real data set has been analyzed to illustrate the flexibility of the proposed distribution.

The paper is organized as follows: In section 2, some preliminaries are provided. In section 3, the proposed bivariate exponentiated Pareto distribution is defined. The estimation of the unknown parameters are presented in section 4. In section 5, we conduct Monte Carlo simulation .

II. Preliminaries

2.1 M mixture representation

The idea of M mixture representation is to write the density of a random variable T on $(0, \infty)$ in the form of compound distribution as follows:

$$f_T(t) = \int_\Omega f_{T|U}(t|u)\, f_U(u) du, \qquad (1)$$

for all $u \in \Omega$, U is a non-negative latent variable follows a gamma distribution with shape parameter 2 and scale parameter 1, denoted by G (2; 1).
And $f_{T|U}(t|u)$ can be written as follows

$$f_{T|U}(t|u) = \frac{h_{T(t)}}{u} I\{u > H_T(t)\}, \quad (2)$$

And

$$f_U(u) = u \exp(-u), u \geq 0. \quad (3)$$

where $h_{T(t)}$ is the hazard function, and $H_T(t)$ is the cumulative hazard function .

In(1999), walker and Stephen used this idea for the Weibull distribution in order to introduce their new family of distributions, called beta-log-normal, by replacing the $G(2,1)$ mixing distribution with the closest log-normal-distribution, $LN(\mu, \sigma^2)$. Adham and Walker (2001)applied this mixture representation to the Gompertz distribution in order to introduce a multivariate Gompertz-type distribution. Furthermore, one may construct a more general lifetime distribution with more flexibility, for example, a generalized exponential, Weibull or Gompertz, via the generalization of the mixing distribution $G(2,1)$. That is, one can use $G(\theta, 1)$ as a mixing distribution, instead of the $G(2,1)$, in order to present a generalized lifetime distribution. Hence, the mixing distribution given by (3) is replaced by,

$$f_U(u) = \frac{1}{\Gamma\theta} u^{\theta-1} \exp(-u), u \geq 0. \quad (4)$$

Adham (2001) has implemented the above mixture representation, with the $G(\theta, 1)$ mixing distribution. She has introduced a generalized Gompertz distribution. In addition, she has discussed Bayesian inference of the generalized Gompertz distribution.

Here, if concentrate on the exponentiated Pareto (EP) distribution, the cumulative distribution function (cdf) is given by:

$$F(t, \theta, \lambda) = \left[1 - (1+t)^{-\lambda}\right]^{\theta}, t > 0, \theta > 0, \lambda > 0, \quad (5)$$

where$\theta$and λ are two shape parameters.

The corresponding probability density function (pdf) is given by

$$f(t, \theta, \lambda) = \theta \lambda \left[1 - (1+t)^{-\lambda}\right]^{\theta-1} (1+t)^{-(\lambda+1)}, t > 0, \theta > 0, \lambda > 0. \quad (6)$$

The hazard function is given by:

$$h(t, \theta, \lambda) = \frac{\theta \lambda \left[1 - (1+t)^{-\lambda}\right]^{\theta-1} (1+t)^{-(\lambda+1)}}{1 - [1 - (1+t)^{-\lambda}]^{\theta}}, t > 0, \theta > 0, \lambda > 0. \quad (7)$$

The cumulative hazard function is given by:

$$H(t, \theta, \lambda) = -ln\left[1 - \left[1 - (1+t)^{-\lambda}\right]^{\theta}\right], t > 0, \theta > 0, \lambda > 0. \quad (8)$$

2.2 Copula

Sklar (1959) introduced the name of "copula". A copula is a function which joins or “couples” a multivariate distribution function to its one-dimensional marginal distribution functions. As suggested by Trivedi and Zimmer (2007) the copula function allows for the construction of previously unknown bivariate distributions derived from known marginals. In addition, this function can link any type of marginal distribution, it is straightforward to construct bivariate distributions using marginals from completely different families while other construct bivariate distributions methods such as conditional distributions and mixing distributions, often rely on marginals from the same family.

A bivariate copula can be defined informally as follows: Let $T_1$and T_2be continuous random variables with distribution functions

$F_{T_1}(t_1) = P(T_1 \leq t_1)$and $F_{T_2}(t_2) = P(T_2 \leq t_2)$.

According to Sklars theorem, there exists a copula C such that

$$F_T(t_1, t_2) = C\left(F_{T_1}(t_1), F_{T_2}(t_2)\right). \quad (9)$$

If $F_{T_1}(t_1)$ and $F_{T_2}(t_2)$ are continuous and differentiable and C is unique then from (3), the joint density can be written as

$$f_T(t_1, t_2) = \prod_{j=1}^{2} f_{T_j}\left(t_j\right) C`\left(F_{T_1}(t_1), F_{T_2}(t_2)\right), \qquad (10)$$

where $f_{T_1}(t_1)$, $f_{T_2}(t_2)$ are the density functions corresponding to $F_{T_1}(t_1), F_{T_2}(t_2)$ and $C` = \frac{\partial^2 C}{\partial F_{T_1} \partial F_{T_2}}$ is the copula density.

A large number of copulas have been proposed in the literature, and each of these imposes a different dependence structure on the data, the interested readers are referred to Joe and Xu (1996), Trivedi and Zimmer (2007), Balakrishnan and Lai (2009) and Nelsen (2013). In this paper, we focus on the Gaussian copula since it is a flexible and has full range of dependence. In addition, it is easy to generalize to multi-dimensions.

The distribution function of bivariate Gaussian (normal) copula with correlation parameter ρ take the form

$$C_{Gauss}(v_1, v_2 ; \rho) = \Phi_\rho(\Phi^{-1}(v_1), \Phi^{-1}(v_2), \rho)$$
$$= \int_{-\infty}^{\Phi^{-1}(v_2)} \int_{-\infty}^{\Phi^{-1}(v_1)} \frac{exp\left\{\frac{-1}{2(1-\rho^2)}(y_1^2 - 2\rho y_1 y_2 + y_2^2)\right\}}{2\pi\sqrt{1-\rho^2}} dy_1 dy_2, \qquad (11)$$

where Φ_ρ denotes the bivariate standard normal distribution function with correlation parameter $\rho \in (-1, 1)$ and Φ^{-1} denotes the inverse of univariate standard normal distribution function.
The density of the bivariate Gaussian copula is

$$C`_{Gauss}(v_1, v_2 ; \rho) = \frac{\exp\left\{\frac{-1}{2(1-\rho^2)}(y_1^2 - 2\rho y_1 y_2 + y_2^2)\right\}}{2\pi\sqrt{1-\rho^2}}, \qquad (12)$$

where $y_1 = \Phi^{-1}(v_1)$, $y_2 = \Phi^{-1}(v_2)$, $v_1 = F_1(t_1)$is the marginal distribution for the random variableT_1 and $v_2 = F_2(t_2)$ is the marginal distribution for the random variableT_2.

III. Bivariate Exponentiated Pareto Distribution Based on Mixture and Gaussian Copula

Adham and Walker (2001) combine the mixing and copula ideas in order to come up with bivariate Gompertz distribution, which is easy to analyse and allows full dependence structures. Their idea consists of constructing a bivariate gamma distribution of the latent variables $\underline{U} = (U_1, U_2)$with two marginal gamma distribution denoted by G(2,1) using copula. That is, obtaining a bivariate gamma distribution with only unknown correlation parameter (ρ). Then obtaining the joint bivariate distribution of U and$\underline{T} = (T_1, T_2)$ where $\underline{T}$ is assumed to be conditionally independent given $\underline{U}$, and then integrate over the latent variables $\underline{U}$ to obtain the required joint distribution of $\underline{T}$.

Now, our bivariate exponentiated Pareto distribution based on mixture and Gaussian copula can be motivated as follows. At the first stage, for j=1,2 assume that $U_j \sim G(2,1)$ with distribution function

$$F_{U_j}(u_j) = \int_0^{u_j} u_j e^{-u_j} du_j = 1 - e^{-u_j}(1 + u_j), \qquad u_j > 0 \qquad (13)$$

By applying Gaussian copula, let $v_j = F_{U_j}(u_j)$. Then, the joint copula distribution function of v_1 and v_2 is given by (9). Hence, from (9), the joint distribution function of the latent (mixing) variables $U_1$and U_2 can be written as

$$F(u_1, u_2) = C_G(v_1, v_2, \rho), \qquad (14)$$

where $C_G(v_1, v_2, \rho)$is the distribution function for Gaussian copula given by (12). Then, from (10) the bivariate gamma density function is given by

$$f_U(u_1, u_2) = f_{U_1(u_1)} f_{U_2(u_2)} \grave{C}_G(v_1, v_2), \qquad (15)$$

where $f_{U_j(u_j)}$is the G(2,1) density function, $v_j = F_{U_j}(u_j)$ is the G(2,1) distribution function given by (13) and $\grave{C}_G(v_1, v_2)$ is given by (12).

The second stage uses the bivariate gamma distribution given by (15) as mixing distribution of T_1, T_2. Assuming that T_1, T_2 are conditionally independent given U.
That is,

$$f_{(\underline{T}|\underline{U})}(t|u) = \prod_{j=1}^{2} f_{(T_j|U_j)}(t_j|u_j), \tag{16}$$

where,

$$f_{(T_j|U_j)}(t_j|u_j) = \frac{h_{T_j}(t_j)}{u_j}\, I\{u_j > H_{T_j}(t_j)\}, \tag{17}$$

and $h_{T_j}(t_j)$, $H_{T_j}(t_j)$ are the hazard and cumulative hazard functions of the univariate EP distribution given by () and () after indexing T, θ and λ by j, j=1,2, respectively.
Hence, the joint density function of T_1, T_2 is given by

$$f_{\underline{T}}(t_1,t_2) = \int_{H(t_2)}^{\infty}\int_{H(t_1)}^{\infty} f_{(\underline{T}|\underline{U})}(t|u)\, f_U(u_1,u_2)\; du_1 du_2, \tag{18}$$

where $f_U(u_1,u_2)$ is given by (15).
In other words, the density function of the BEPII can be written as

$$f_T(t_1,t_2) = \int_{H(t_2)}^{\infty}\int_{H(t_1)}^{\infty}\prod_{j=1}^{2}\frac{h_{T_j}(t_j)}{u_j} f_{U_j(u_j)}\grave{C}_G(v_1,v_2)\, du_1 du_2,$$

$$= \int_{H(t_2)}^{\infty}\int_{H(t_1)}^{\infty}\prod_{j=1}^{2}\frac{\theta_j\,\lambda_j\left[1-(1+t_j)^{-\lambda_j}\right]^{\theta_j-1}(1+t_j)^{-(\lambda_j+1)}}{1-\left[1-(1+t_j)^{-\lambda_j}\right]^{\theta_j}}\, e^{-u_j}\,\grave{C}_G(v_1,v_2)\, du_1 du_2, \tag{19}$$

where $H(t_j)$is the cumulative hazard function of the univariate EP distribution given by (8) and $\grave{C}_G(v_1,v_2)$ is given by (12).
Construct a bivariate exponentiated Pareto distribution (or any other life time distribution) using above two stages method will help in the model analysis, because we can estimate the correlation parameter ρ from the first stage (i.e the bivariate gamma distribution). Then, estimate the other parameters $\theta_1,\ \lambda_1,\ \theta_2\ and\ \lambda_2$ from the second stage (i.e the conditional density functions $f_{(T_j|U_j)}(t_j|u_j)$).

IV. Estimation

4.1 Maximum Likelihood Estimation

In this subsection, we discuss the ML estimation of the unknown parameters of the BEPII distribution. Let $T_i = (T_{1i}, T_{2i})^{`}, i = 1,..,n$ be a bivariate random sample of size n from BEPII distribution given by (3.14). If $\underline{T}$ is assumed to be conditionally independent given $\underline{U}$, and $\underline{U} = (U_1, U_2)^{`}$ is latent variables from bivariate gamma distribution based on Gaussian copula.
The two-stage method can be incorporated as follows.
First: compute the estimates of the marginal parameters $(\theta_1,\lambda_1,\theta_2,\lambda_2)$ by maximizing $l(\theta_1,\lambda_1,\theta_2,\lambda_2)$ with respect to $\theta_1,\lambda_1,\theta_2$ and λ_2.

$$l(\theta_1,\lambda_1,\theta_2,\lambda_2,u_1,u_2) = \sum_{i=1}^{n}\sum_{j=1}^{2}\log f_{T|U}(t_i,u_i)$$

$$= \sum_{j=1}^{2}\left\{n log\theta_j + n log\lambda_j + (\theta_j-1)\sum_{i=1}^{n} log\left(1-(1+t_{ji})^{-\lambda_j}\right) - (\lambda_j+1)\sum_{i=1}^{n} log\,(1+t_{ji})\right.$$
$$\left. - \sum_{i=1}^{n} log u_{ji} - \sum_{i=1}^{n} log\,[1-\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j}\right\} \tag{20}$$

Second: obtain the estimate of correlation parameter $(\rho)$of bivariate gamma distribution based on Gaussian copula by maximizing

$$l(\rho) = \sum_{i=1}^{n}\log C_G\,(v_{i1},v_{i2},\rho), \tag{21}$$

where v_{ij} is the CDF of G(2,1), for j = 1,2

4.2 Bayesian Estimation

The Gibbs sampler procedure is used to obtain Bayesian estimates of the unknown parameters of the BEPII distribution. Assume that the prior distributions of the parameters are

$$\pi(\theta_j) \propto \frac{1}{\theta_j}, \; and \pi(\lambda_j) \propto \frac{1}{\lambda_j}, \qquad j = 1,2 \qquad (22)$$

and $\pi(\rho)$ has a uniform prior distribution defined on the interval (-1, 1).

Let $T_i = (T_{1i}, T_{2i})`, i = 1,2, \dots n$ be a bivariate random sample of size n from BEPII distribution where $\underline{T}$ is assumed to be conditionally independent given $\underline{U}$, and

$\underline{U} = (U_1, U_2)`$, latent variables from bivariate gamma distribution based on Gaussian copula.

Therefore, the joint posterior distribution is given by

$$f(\rho,\theta_1,\theta_2,\lambda_1,\lambda_2,\underline{u}|\underline{t}) = \prod_{j=1}^{2}\{\pi(\theta_j)\pi(\lambda_j)\}\pi(\rho)\prod_{i=1}^{n} f_{\underline{T},\underline{U}}(t_i,u_i), (23)$$

where

$$f_{\underline{T},\underline{U}}(t_i,u_i) = \prod_{j=1}^{2}\left(\frac{h(t_{ji})}{u_{ji}}\right) f_{U_{ji}}(u_{ji}) C_G(v_1,v_2) I\{u_{ji} > H(t_{ji})\},$$

Where $h_{T_j}(t_j)$, $H_{T_j}(t_j)$ are the hazard and cumulative hazard functions of the univariate EP distribution given by (7) and (8) after indexing T, θ and λ by j, j=1,2, respectively, and $f_{U_{ji}}(u_{ji})$ is the PDF of G(2,1).

The full conditional distributions of the Gibbs sampler are

1 Sample uji from $f(u_{ji}|\theta_1,\theta_2,\lambda_1,\lambda_2,u_{-ji},\underline{t})$ where, i=1,…,n, and $j \neq -j, j = 1,2$

$$f(u_{ji}|\theta_1,\theta_2,\lambda_1,\lambda_2,u_{-ji},\underline{t}) \propto exp\left\{-u_{ji} - \frac{x_i}{2}\right\} I\left\{u_{ji} > -ln\left[1 - \left[1 - (1+t_{ji})^{-\lambda_j}\right]^{\theta_j}\right]\right\}$$

Where $x_i = \frac{y^2{}_{1i} - 2\rho y_{1i} y_{-2i} + y^2{}_{2i}}{1-\rho^2}$ .we use the following steps to sample u_{ji} from $f(u_{ji}|\theta_1,\theta_2,\lambda_1,\lambda_2,u_{-ji},\underline{t})$

a-Introduce a non-negative latent variable τ, such that

$$f(u_{ji},\tau) \propto e^{-u_{ji}} I\left\{\tau < e^{-\frac{x_i}{2}}\right\} I\left\{u_{ji} > -ln\left[1 - \left[1 - (1+t_{ji})^{-\lambda_j}\right]^{\theta_j}\right]\right\}$$

b- Choose the initial values of u_{ji} to be

$$u_{ji} = -ln\left[1 - \left[1 - (1+t_{ji})^{-\lambda_j}\right]^{\theta_j}\right] + 1$$

c- Sample τ from Uniform $\left(0, exp\left(-\frac{x_i}{2}\right)\right)$

d- Sample u_{ji} from $f(u_{ji}|\tau)$, where

$f(u_{ji}|\tau) \propto e^{-u_{ji}} I(A_i < u_{ji} < B_i)$

where

$$A_i = max\left\{-ln\left[1 - \left[1 - (1+t_{ji})^{-\lambda_j}\right]^{\theta_j}\right], F_{U_j}^{-1}[\Phi(\delta_{1i})]\right\}. \qquad (24)$$

$$B_i = F_{U_j}^{-1}[\Phi(\delta_{2i})]. \qquad (25)$$

$\delta_{1i} = (\rho y_{-ji}) + q_i, \qquad \delta_{2i} = (\rho y_{-ji}) - q_i,$

and

$$q_i = \sqrt{-2(1-\rho^2)\left[ln(\tau) + \frac{y^2_{-ji}}{2}\right]}$$

$f(u_{ji}|\tau)$ is a double truncated distribution that can be sampled easily by using the inverse distribution function method. Then, for $V \sim Uniform\ (0,1)$

$u_{ji} = -ln[e^{-A_i} - V(e^{-A_i} - e^{-B_i})]$ (26)

where A_i and B_i are given by (24) and (25), respectively.

2 Sample λ_j from $f(\lambda_j|\lambda_{-j},\theta_1,\theta_2,\underline{u},\underline{t})$,where $j \neq -j, -j, j = 1,2$.we use the following steps to sample λ_j from $f(\lambda_j|\lambda_{-j},\theta_1,\theta_2,\underline{u},\underline{t})$

$$f(\lambda_j|\lambda_{-j},\theta_1,\theta_2,\underline{u},\underline{t}) \propto \lambda_j^{n-1} exp\left\{\sum_{i=1}^{n} ln\left[\frac{\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j - 1}(1+t_{ji})^{-(\lambda_j+1)}}{1-\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j}}\right]\right\} I\{\lambda_j < A_j\}$$

where

$$A_j = min\left[\frac{-ln\left(1-(1-e^{-u_{ji}})^{\frac{1}{\theta_j}}\right)}{ln(1+t_{ji})}\right] \quad (27)$$

a-Introduce a non-negative latent variable v, such that
$f(\lambda_j, v) \propto \lambda_j^{n-2}\, \mathrm{I}\,(v < \lambda_j d_j)\, \mathrm{I}\{\lambda_j < A_j\}$,
where

$$d_j = exp\left\{\sum_{i=1}^{n} ln\left[\frac{\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j-1}(1+t_{ji})^{-(\lambda_j+1)}}{1-\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j}}\right]\right\}$$

b- Choose the initial values of λ_j
c- Sample v from U(0, $\lambda_j d_j$)
d-Sample λ_j from $f(\lambda_j|v)$ where
$f(\lambda_j|v) \propto \lambda_j^{n-2}\mathrm{I}(B_j < \lambda_j < A_j)$,
Where A_j is given by (27)

$$b_j = \frac{v}{d_j}. \quad (28)$$

$f(\lambda_j|V)$ can be sampled easily by using the inverse distribution function method. Then, for $\delta \sim U(0,1)$
$\lambda_j = \left[(A_j^{n-1} - B_j^{n-1})\delta + B_j^{n-1}\right]^{1/n-1}$,
where A_j and B_j are given by (27) and (28), respectively.
3) Sample θ_j from $f(\theta_j|\theta_{-j}, \lambda_1, \lambda_2, \underline{u}, \underline{t})$,where $j \neq -j, -j, j = 1,2$.we use the following steps to sample θ_j from $f(\theta_j|\theta_{-j}, \lambda_1, \lambda_2, \underline{u}, \underline{t})$

$$f(\theta_j|\theta_{-j}, \lambda_1, \lambda_2, \underline{u}, \underline{t}) \propto \theta_j^{n-1} exp\left\{\sum_{i=1}^{n} ln\left[\frac{\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j-1}}{1-\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j}}\right]\right\} I\{\theta_j < A_j\}.$$

where $A_j = max\left[\frac{ln(1-e^{-u_{ji}})}{ln\left(1-(1+t_{ji})^{-\lambda_j}\right)}\right]$ (29)

a- Introduce a non-negative latent variable v, such that
$f(\theta_j, v) \propto \theta_j^{n-2}\mathrm{I}(v < \theta_j d_j)\mathrm{I}(\theta_j < A_j)$,
where $d_j = exp\left\{\sum_{i=1}^{n} ln\left[\frac{\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j-1}}{1-\left(1-(1+t_{ji})^{-\lambda_j}\right)^{\theta_j}}\right]\right\}$.
b- Choose the initial values of θ_j
c- Sample v from U(0, $\theta_j d_j$)
d- Sample θ_j from $f(\theta_j|v)$ where
$f(\theta_j|v) \propto \theta_j^{n-2}\mathrm{I}(B_j < \theta_j < A_j)$,
where A_j is given by (27),

$$B_j = \frac{v}{d_j}. \quad (30)$$

$f(\theta_j|v)$ can be sampled easily by using the inverse distribution function method. Then, for $\delta \sim U(0,1)$

$$\theta_j = \left[(A_j^{n-1} - B_j^{n-1})\delta + B_j^{n-1}\right]^{1/n-1}, \quad (31)$$

where A_j and B_j are given by (29) and (30), respectively.
4) Sample ρ from its posterior distribution

$$f(\rho|\theta_1, \theta_2, \lambda_1, \lambda_2, \underline{u}, \underline{t}) \propto (1-\rho^2)^{\frac{-n}{2}} exp\left\{\sum_{i=1}^{n} \frac{y^2_{1i} - 2\rho y_{1i} y_{2i} + y^2_{2i}}{-2(1-\rho^2)}\right\},$$

To sample the above posterior distribution of ρ, the following accept-reject algorithm is motivated:
Sample ρ^* from Uniform (-1,1) and v from Uniform (0,1).
If $v \le \frac{f(\rho^*|\underline{\theta}, \underline{\lambda}, \underline{u}, \underline{t})}{f(\hat{\rho}|\underline{\theta}, \underline{\lambda}, \underline{u}, \underline{t})}$, where $\hat{\rho} = \sum_{i=1}^{n} \frac{y_{1i} y_{2i}}{n}$, the maximum likelihood estimate of ρ, then accept ρ^* (i.e. $\rho = \rho^*$), otherwise repeat steps a and b.

V. Simulation

In this section, simulation study have been performed for different sample sizes n=15, 30, 50, 80, 100 and 150 and for different values of the copula parameter, keeping $\theta_1 = \theta_2 = .9$ and $\lambda_1 = \lambda_2 = 1.8$ and $\rho = 0.5, 0.7$ and 0.8. Since we observed the performances to be quite similar for negative ρ, so we present the results only for positive ρ. In each case we have obtained ML and Bayesian parameter estimates for the BEPII distribution using the following:

Algorithm (1): Generating a random samples from BEPII distribution

The following steps are used to generate random samples from BEPII distribution:

1. Generate latent random sample from the bivariate gamma distribution based on Gaussian copula.
2. Generate n independents U(0,1) random variables.
3. For a given values of the parameters $(\theta_1, \lambda_1, \theta_2, \lambda_2, \rho)$, the inverse distribution function method can be used to generate samples from BEPII distribution using the $f_{(T_j|U_j)}(t_j|u_j)$inequation (17).

Steps for obtaining the ML estimation for the parameters of BEPII distribution:

The ML estimates of the parameters are obtained numerically according to the following steps:

1. For given value of the parameters of $({\theta_1}^*, {\lambda_1}^*, {\theta_2}^*, {\lambda_2}^*, \rho^*)$ a sample of size n from BEPII distribution is generated using Algorithm (1).
2. The ML estimates of the marginal parameters is obtained by maximizing (20) with respect to $\theta_1, \lambda_1, \theta_2$ and λ_2.
3. The ML estimate of the correlation parameter ρ of the bivariate gamma distribution based on Gaussian copula is obtained by maximizing (21) with respect to ρ

Steps for obtaining the Bayesian estimation for the parameters of BEPII distribution:

The Bayesianestimates of the parameters are obtained numerically according to the following steps:

1- For given value of the parameters $({\theta_1}^*, {\lambda_1}^*, {\theta_2}^*, {\lambda_2}^*, \rho^*)$, a sample of size n from BEPII distribution is generated using Algorithm (1).

2- Generate a bivariate observations (y_1, y_2), using the formula

$$y_j = \Phi^{-1}\left\{F_{U_j}(u_j)\right\}$$

where $\left\{F_{U_j}(u_j)\right\}$is the distribution function of Gamma (2, 1), given by (13) , Φ^{-1} denotes the inverse of univariate standard normal distribution function.

3- The Bayesian estimates of ρ is computed using the Metro_Hasting function in the (MHadaptive) package and the following are defined:

- The log-likelihood function and the prior density function of the correlation parameter ρ in (22) and (23)
- The initial value of ρ
- The number of iteration to run the chain.

For 1000 replications, ML and Bayesian estimates of the parameters along with the bias and MSE are reported in Table (1).

The results of table (1) for the ML and Bayesian estimation of the unknown parameters can be summarized as follows:

- The performances of the ML and Bayesian estimates are quite satisfactory. It is observed that when the sample size increases, the MSE decrease for all the parameters, as expected. In addition, as sample size increases the bias for all parameters is mostly decreasing.
- The bias and MSE of ML and Bayesian estimation of marginals parameters do not seem to depend on ρ.
- In this subsection,the performances of the ML and Bayesian estimators are compared based on the MSE through Monte Carlo simulations and the results are reported in Table (1).

Table (1): The average estimates and the corresponding MSE (reported within brackets) of the ML and Bayesian estimators under BEPII distribution when $\theta_1 = \theta_2 = 0.9, \lambda_1 = \lambda_2 = 1.8$ with different values of ρ

n	method	ρ	$\hat{\theta}_1$	$\hat{\lambda}_1$	$\hat{\theta}_2$	$\hat{\lambda}_2$	$\hat{\rho}$
15	ML	0.5	1.1347(0.2473)	2.1748(0.7366)	1.1347(0.2973)	2.1528(0.7357)	0.4853(0.0392)
		0.7	1.1361(0.2479)	2.1705(0.7195)	1.1350(0.2909)	2.1483(0.6779)	0.6853(0.0192)
		0.8	1.1367(0.2511)	2.1672(0.7069)	1.1359(0.2910)	2.1470(0.6568)	0.7880(0.0100)
	Bayesian	0.5	0.7070(0.0397)	1.8135(0.0195)	0.7087(0.0389)	1.8141(0.0194)	0.2675(0.0974)
		0.7	0.7129(0.0374)	1.8136(0.0195)	0.7148(0.0366)	1.8143(0.0195)	0.5089(0.0681)
		0.8	0.7171(0.0359)	1.8137(0.0195)	0.7190(0.0351)	1.8144(0.0195)	0.6565(0.0405)
30	ML	0.5	1.0126(0.1189)	2.0061(0.3229)	0.9750(0733)	1.9257(0.2483)	0.5041(0.0219)
		0.7	1.0116(0.1297)	1.9998(0.3147)	0.9792(0.0818)	1.9327(0.2470)	0.6992(0.0107)
		0.8	1.0303(0.1193)	2.0240(0.3263)	1.0068(0.0953)	1.9798(0.2813)	0.8001(0.0051)
	Bayesian	0.5	0.7805(0.0151)	1.8044(0.0043)	0.7802(0.0151)	1.8044(0.0043)	0.4486(0.0146)
		0.7	0.7859(0.0139)	1.8045(0.0043)	0.7858(0.0138)	1.8045(0.0043)	0.6353(0.0114)
		0.8	0/7897(0.0130)	1.8046(0.0043)	0.7897(0.0130)	1.8045(0.0043)	0.7450(0.0071)
50	ML	0.5	0.9403(0.0320)	1.9112(0.1508)	0.9224(0.0286)	1.8726(0.1540)	0.5083(0.0130)
		0.7	0.9376(0.0316)	1.9059(0.1499)	0.9231(0.0290)	1.8743(0.1520)	0.7034(0.0061)
		0.8	0.9359(0.0314)	1.9025(0.1494)	0.9238(0.0294)	1.8760(0.1509)	0.8016(0.0031)
	Bayesian	0.5	0.8210(0.0066)	1.8032(0.0016)	0.8211(0.0065)	1.8033(0.0016)	0.4731(0.0087)
		0.7	0.8255(0.0059)	1.8033(0.0016)	0.8255(0.0059)	1.8033(0.0016)	0.6693(0.0048)
		0.8	0.8286(0.0055)	1.8034(0.0016)	0.8287(0.0054)	1.8034(0.0016)	0.7749(0.0026)
100	ML	0.5	0.9170(0.0151)	1.8441(0.0649)	0.9141(0.0139)	1.8406(0.0603)	0.5014(0.0061)
		0.7	0.9167(0.0149)	1.8440(0.0643)	0.9141(0.0139)	1.8406(0.0601)	0.6998(0.0028)
		0.8	0.9164(0.0148)	1.8438(0.0638)	0.9141(0.0139)	1.8407(0.0601)	0.7995(0.0004)
	Bayesian	0.5	0.8563(0.0020)	1.8004(0.0003)	0.8560(0.0020)	1.8004(0.0003)	0.5128(0.0040)
		0.7	0.8590(0.0017)	1.8004(0.0003)	0.8587(0.0018)	1.8004(0.0003)	0.6996(0.0017)
		0.8	0.8609(0.0016)	1.8004(0.0003)	0.8606(0.0016)	1.8004(0.0003)	0.7975(0.0008)

From Table (1), in general, the MSE of the Bayesian estimates of the parameters smaller than their corresponding MSE of ML estimates. However, it observed that for small sample size, the MSE of ML estimate of ρ smaller than their corresponding MSE of Bayesian estimate. Therefore, it can be concluded that the Bayesian method based on non-informative prior perform better than the ML method.

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