IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 6 Ver. IV (Nov. - Dec. 2017), PP 65-66
www.iosrjournals.org

Analyticity of Rank of Operators on A Banach Space

[1]Nahid H. k. Abdelgader
Corresponding Author: Nahid H. k. Abdelgader:

ABSTRACT. If $G(z)$ is an analytic family of operators on a Banach space which is of finite rank for each z, then rank $G(z)$ is constant except for isolated points.

In this note we consider the analytic group $G(z)$ of operators on a complex Banach space x, such that the rank of $G(z)$ is finite for each z . We show that the rank of $G(z)$ is constant on the domain of analyticity, unless for seperated points. .

Definition 1 *Let X be a real vector space . The complexfication of X is the complex vector space $X_{\mathbb{C}} := X \otimes \mathbb{C}$, with scalar multiplication defined by $\alpha(x \otimes \beta) := x \otimes \alpha\beta(\alpha\beta \in \mathbb{C})$*

Lemma 1 *If $G \in \mathbb{B}(x)$, then rank $G \geqslant N$ iff there exist bounded projections P and$Q$ of dimension N such that PFQ has rank N.*

Proof. If rank $G < N$, then rank $PGQ \leqslant rankG < N$. Conversely, if $rankG \geqslant N$, there are $X_1, \ldots, x_N$ such that$Gx_1, \ldots, Gx_N$ are linearly independent. If P projects on the span of $Gx_1, \ldots, Gx_N$ and Q on the span of $X_1, \ldots, x_N$, then PGQ has rank N.
Now we show that If $G(z)$ is analytic on a domain Ω and rank $G(z)$ is finite for each z, then there is an integer n such that rank $G(z) = n$ except at some points where $n \geq rankG(z)$.
Proof For each $k \leqslant 0$, let $E_{j-1} = \{z \in \Omega | rankG(z) \leqslant j-1\}$. Since $\Omega = \bigcup_j^\infty E_{j-1}, E_{j-1}$ is uncountable for some integer$k$, and so there is a smallest integer n such that E_n has a point of accumulation within Ω .
If P and Q are arbitrary projections with dim $P = dimQ > n$, then the determinant $d(z)$ of $PG(z)Q$, computed with respect to fixed bases of Px and Qx, vanishes on E_n, and hence on all of Ω. Since P and Q are arbitrary, the follwing lemma satisfying $E_n = \Omega$. Since n is minimal, $E_(n-1)$ consists of isolated points.

This proof also shows that the rank of $G(z)$ is determined by its values on any set with an accumulation point in Ω , and hence that no analytic continuation of $G(z)$ can have rank exceeding n.
When we refer to the lemma we find that the norm and $rankG_n \leqslant n$, then $rankG \leqslant n$. For if P and Q have the same dimension exceeding m , then $detPGQ = \lim \det PG_nQ = 0$ .The hypothesis of previous theorem can be weakened by assuming only that the set of points at which $G(z)$ has finite rank is uncountable; however, it does not suffice to assume only that $G(z)$ has finite

rank on a set with an accumulation point in Ω, for if $G(z)$ is the infinite diagonal matrix $G(z)$ with diagonal elements $a_1(z), a_2(z), a_3(z), \ldots a_m(z)$

where $a_m(z) = (z-1)(z-1/2)\ldots(z-1/m)$, then $G(z)$ is analytic for $\mid z \mid < 1$, while $rankG(1/n) = m-1$.

If $G \in \mathcal{B}(X)$ has finite rank, then we let $\beta(G)$ denote the operator norm of G and

$$\tau(G) = \inf \sum_{i=1}^{n} \mid x_i^* \mid\mid x_i \mid$$

where the infimum is taken over all representations $G = \inf \sum_{i=1}^{n} \langle x_i^*, \cdot \rangle x_i$ of G. τ is a norm, and

$$\mid trG \leq \tau(F), \tag{1}$$

$$B(G) \leq \tau(G) \leq \beta(G)\ rank\ G \tag{2}$$

and

$$\tau(AG) \leq B(A)\tau(G) for\ any\ A\ in\ B(x). \tag{3}$$

Theorem if $G(z)$ is analytic and the rank of $G(z)$ is finite for all z in Ω , then $trG(z)$ is analytic, and $tr\frac{G(z)}{dz} = trG'(z)$

Proof. the rank $G(z) \leqslant n \leq \infty$ for some integer n. The rank of $D(z,h) = h^{-1}[F(z+h) - F(z)]$ cannot exceed $2n$, so that $midh^-1[trG(z+h) - trG(z) - trG'(z)]$

$= midtr(D(z,h) - G'(z)) \leqslant \tau(D(z,h) - G'(z))$

$\leqslant 4n\beta(D(z,h) - G'(z))$.

But the final term tends to zero as $h \longrightarrow 0$, since $G(z)$ is analytic in norm.

References

[1]. J.M.A.M. van Neerven ,The Norm of a Complex Banach Lattice,Department of Mathematics TU Delft .

[2]. JAMES S. HOWLAND ,Analyticity of Determinants of operators on a Ba- nach Space.

Nahid H. k. Abdelgader "Analyticity of Rank of Operators on A Banach Space" IOSR Journal of Mathematics (IOSR-JM) 13.6 (2017): 65-66.