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Differential Geometry of Manifolds, Surfaces and Curves.

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Abstract: *To study problems in geometry the technique known as Differential geometry is used. Through which in calculus, linear algebra and multi linear algebra are studied from theory of plane and space curves and of surfaces in the three-dimensional Euclidean space formed the basis for development of differential geometry during the 18th century and the 19th century. Since the late 19th century, differential geometry has grown into a field concerned more generally with the geometric structures on differentiable manifolds. The differential geometry of surfaces captures many of the key ideas and techniques characteristic of this field.*
Keywords: *Curvature Manifolds, Riemannian geometry and surface of revolutions.*

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I. Introduction

Carl Friedrich Gaul (1777-1855)[1] is the father of differential geometry. He was (among many other things) a cartographer and many terms in modern differential geometry (chart, atlas, map, coordinate system, geodesic, etc.) reflect these origins. He was led to his Theorema Egregium by the question of whether it is possible to draw an accurate map of a portion of our planet.

We can distinguish extrinsic differential geometry and intrinsic differential geometry. The former restricts attention to sub manifolds of Euclidean space while the latter studies manifolds equipped with a Riemannian metric. The extrinsic theory is more accessible because we can visualize curves and surfaces in R 3, but some topics can best be handled with the intrinsic theory.

Organization of the paper is with respective sections: manifold, discussion of major branches of differential geometry, applications of differential geometry, differential geometry of curvature, differential geometry of surfaces and conclusions.

II. Manifold

In differential geometry, a differentiable manifold is a space which is locally similar to a Euclidean space [2]. In an n-dimensional Euclidean space any point can be specified by n real numbers. These are called the coordinates of the point. An n-D differentiable manifold is a generalization of n-dimensional Euclidean space. In a manifold it may only be possible to define coordinates locally. This is achieved by defining coordinate patches: subsets of the manifold which can be mapped into n-dimensional Euclidean space.

1.1 Kähler manifold

Kähler manifold is three mutually compatible structures; a complex structure, a Riemannian structure, and a symplectic structure. It finds important applications in the field of algebraic geometry where they represent generalizations of complex projective algebraic varieties via the Kodaira embedding theorem

Definition 2.1.1: Symplectic viewpoint: A Kähler manifold is a symplectic manifold (K, ω) equipped with an integral almost-complex structure which is compatible with the symplectic form.

Definition 2.1.2: Complex viewpoint: A Kähler manifold is a Hermitian manifold whose associated Hermitian form is closed. The closed Hermitian form is called the Kähler metric.

Definition 2.1.3: Equivalence: Every Hermitian manifold $\boldsymbol{K}$ is a complex manifold which comes naturally equipped with a Hermitian form $\boldsymbol{h}$ and an integral, almost complex structure $\boldsymbol{J}$. Assuming that $\boldsymbol{h}$ is closed, there is acanonical symplectic form defined as $\boldsymbol{\omega} = \frac{i}{2}(\boldsymbol{h} - \overline{\boldsymbol{h}})$ which is compatible with J, hence satisfying the first definition. On the other hand, any symplectic form compatible with an almost complex structure must be a complex differential form of type (1, 1), written in a coordinate chart (U, Z_i) as $\omega = \frac{i}{2}\sum_{j,k} h_{jk}\, dZ_j \wedge d\bar{Z}_k$ for $h_{jk} \in c^{\infty}(U, C)$. The added assertions that ω be real-valued, closed, and non-degenerate guarantee h_{jk} that defines Hermitian forms at each point in K.

Remark 2.1.4: Relation between Hermitian and symplectic definitions
Let h be the Hermitian form, ω the symplectic form, and J, the almost complex structure. Since ω and J are compatible, the new form $g(u, v) = \omega(u, Jv)$ is Riemannian. One may then summarize the connection between these structures via the identity $h = g + i\omega$

1.2 Laplacians on Kähler manifolds

Definition 2.2.1: Let * be the Hodge operator and then on an differential manifold *X* we can define the Laplacian as $\Delta_d = dd^* + d^*d$ Where d is the exterior derivative and $d^* = -(-1) * d *$.Furthermore if *X* is Kähler then d and d^* are decomposed as $d = \partial + \bar{\partial}$, $d^* = \partial^* + \bar{\partial}^*$ and we can define another Laplacians $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, $\Delta_{\partial} = \partial\partial^* + \partial^*\partial$ that satisfy $\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$. From these facts we obtain the Hodge decomposition $H^r = \bigoplus_{p+q=r}^{H^{p,q}}$ where H^r is r - degree harmonic form and $H^{p,q}$ is {p, q}-degree harmonic form on *X*.

Remark 2.2.2: A differential form $\propto$ is harmonic if and only if each $\propto^{i,j}$ belong to the {i, j}-degree harmonic form.

Definition2.2.3: A pseudo-Riemannian manifold (M, g) is a differentiable manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g which, unlike a Riemannian metric, need not be positive-definite, but must be non-degenerate. Such a metric is called a pseudo-Riemannian metric and its values can be positive, negative or zero. The signature of a pseudo-Riemannian metric is (p, q) where both p and q are non-negative.

Definition2.2.4: Lorentzian manifold: A Lorentzian manifold is an important special case of a pseudo-Riemannian manifold in which the signature of the metric is (1, n−1) (or sometimes (n−1, 1), see sign convention). Such metrics are called Lorentzian metrics. They are named after the physicist Hendrik Lorentz.

III. Discussion Of Major Branches Of Differential Geometry

1.3 Riemannian geometry

It studies Riemannian manifolds, smooth manifolds with a Riemannian metric. This is a concept of distance expressed by means of a smooth positive definite symmetric bilinear form defined on the tangent space at each point. Riemannian geometry generalizes Euclidean geometry to spaces that are not necessarily flat, although they still resemble the Euclidean space at each point infinitesimally, i.e. in the first order of approximation. Various concepts based on length, such as the arc length of curves, area of plane regions, and volume of solids all possess natural analogues in Riemannian geometry. The notion of a directional derivative of a function from multivariable calculus is extended in Riemannian geometry to the notion of a covariant derivative of a tensor. Many concepts and techniques of analysis and differential equations have been generalized to the setting of Riemannian manifolds.A distance-preserving diffeomorphism between Riemannian manifolds is called an isometry. This notion can also be defined locally, i.e. for small neighborhoods of points. Any two regular curves are locally isometric. In higher dimensions, the Riemann curvature tensor is an important point wise invariant associated to a Riemannian manifold that measures how close it is to being flat. An important class of Riemannian manifolds is the Riemannian symmetric spaces, whose curvature is not necessarily constant. These are the closest analogues to the "ordinary" plane and space considered in Euclidean and non-Euclidean geometry.

1.4 Pseudo-Riemannian geometry
Pseudo-Riemannian geometry generalizes Riemannian geometry to the case in which the metric tensor need not be positive-definite. A special case of this is a Lorentzian manifold, which is the mathematical basis of Einstein's general relativity theory of gravity.

1.5 Finsler geometry
Finsler geometry has the Finsler manifold as the main object of study. This is a differential manifold with a Finsler metric, i.e. a Banach norm defined on each tangent space. A Finsler metric is a much more general structure than a Riemannian metric.

Definition: A Finsler structure on a manifold M is a function $F : TM \rightarrow [0, \infty)$ such that:

$F(x,\ my) = |m| F(x, y)$ for all x, y in TM,

F is infinitely differentiable in $TM - \{0\}$,

The vertical Hessian of F^2 is positive definite.

1.6 Symplectic geometry
Symplectic geometry is the study of symplectic manifolds. An almost symplectic manifold is a differentiable manifold equipped with a smoothly varying non-degenerate skew-symmetric bilinear form on each tangent space, i.e., a non degenerate 2-form ω, called the symplectic form. A symplectic manifold is an almost symplectic manifold for which the symplectic form ω is closed: $d\omega = 0$.

Definition: A diffeomorphism between two symplectic manifolds which preserves the symplectic form is called a **symplectomorphism**.

Non-degenerate skew-symmetric bilinear forms can only exist on even dimensional vector spaces, so symplectic manifolds necessarily have even dimension. In dimension 2, a symplectic manifold is just a surface endowed with an area form and a symplectomorphism is an area-preserving diffeomorphism.

1.7 Complex and Kähler geometry
Definition: A real manifold M, endowed with a tensor of type $(1, 1)$, i.e. a vector bundle endomorphism (called an almost complex structure) $J: TM \rightarrow TM \ni J^2 = -1$. It follows from this definition that an almost complex manifold is even dimensional.

Definition: An almost complex manifold is called complex if $N_J = 0$, where N_J is a tensor of type $(2, 1)$ related to J , called the Nijenhuis tensor (or sometimes the torsion).

Remark: An almost complex manifold is complex if and only if it admits a holomorphic.

Definition: An almost Hermitian structure is given by an almost complex structure J, along with a Riemannian metric g, satisfying the compatibility condition $g(JX, JY) = g(X, Y)$

Definition: An almost Hermitian structure defines naturally a differential two-form $\omega_{J,g}(X, Y) = g(JX, Y)$

The following two conditions are equivalent:

1. $N_J = 0;\ d\omega = 0$

2. $\nabla J = 0$ where ∇ is the Levi-Civita connection of g. In this case, (J, g) is called a Kähler structure, and a Kähler manifold is a manifold endowed with a Kähler structure. In particular, a Kähler manifold is both a complex and a symplectic manifold. A large class of Kähler manifolds (the class of Hodge manifolds) is given by all the smooth complex projective varieties.

1.8 CR geometry
CR geometry is the study of the intrinsic geometry of boundaries of domains in complex manifolds.

1.9 Synthetic differential geometry
Synthetic differential geometry is a reformulation of differential geometry in the language of topos theory, in the context of an intuitionistic logic characterized by a rejection of the law of excluded middle. There are several insights that allow for such a reformulation. The first is that most of the analytic data for describing the class of smooth manifolds can be encoded into certain fibre bundles on manifolds: namely bundles of jets . The second insight is that the operation of assigning a bundle of jets to a smooth manifold is functorial in nature. The third insight is that over a certain category, these are representable functors. Furthermore, their representatives are related to the algebras of dual numbers, so that smooth infinitesimal analysis may be used.Synthetic differential geometry can serve as a platform for formulating certain otherwise obscure or confusing notions from differential geometry. For example, the meaning of what it means to

benatural (or invariant) has a particularly simple expression, even though the formulation in classical differential geometry may be quite difficult.

1.10 Abstract differential geometry

The adjective abstract has often been applied to differential geometry before, but the abstract differential geometry (ADG) of this article is a form of differential geometry without the calculusnotion of smoothness, developed by Anastasios Mallios and others from 1998 onwards. Instead of calculus, an axiomatic treatment of differential geometry is built via sheaf theory and sheaf cohomology using vector sheaves in place of bundles based on arbitrary topological spaces. Mallios says non commutative geometry can be considered a special case of ADG, and that ADG is similar to synthetic differential geometry.

1.11 Discrete differential geometry

Discrete differential geometry is the study of discrete counterparts of notions in differential geometry. Instead of smooth curves and surfaces, there are polygons, meshes, and simplicial complexes. It is used in the study of computer graphics and topological combinatorics.

IV. Applications Of Differential Geometry

In physics:

a) Differential geometry is the language in which Einstein's general theory of relativity is expressed. According to the theory, the universe is a smooth manifold equipped with a pseudo-Riemannian metric, which describes the curvature of space-time. Understanding this curvature is essential for the positioning of satellites into orbit around the earth. Differential geometry is also indispensable in the study of gravitational lensing and black holes.

b) Differential forms are used in the study of electromagnetism.

c) Differential geometry has applications to both Lagrangian mechanics and Hamiltonian mechanics. Symplectic manifolds in particular can be used to study Hamiltonian systems.

d) Riemannian geometry and contact geometry have been used to construct the formalism of geometro thermodynamics which has found applications in classical equilibrium thermodynamics.

In economics [2]: differential geometry has applications to the field of econometrics.

Geometric modeling (including computer graphics) and computer-aided geometric design draw on ideas from differential geometry.

In engineering, differential geometry can be applied to solve problems in digital signal processing.

In **probability,** statistics, and information theory, one can interpret various structures as Riemannian manifolds, which yields the field of information geometry, particularly via the Fisher information metric.

In **structural geology,** differential geometry is used to analyze and describe geologic structures.

In **computer vision**, differential geometry is used to analyze shapes.

In **image processing** [3], differential geometry is used to process and analyse data on non-flat surfaces.

In **wireless communications** [4], Grassmanian manifold is used for beam forming techniques in multiple antenna systems.

V. Differential Geometry Of Curvature[5]

1.12 Pinched sectional curvature

1. **Sphere theorem**. : If M is a simply connected compact n-dimensional Riemannian manifold with sectional curvature strictly pinched between 1/4 and 1 then M is diffeomorphic to a sphere.

2. **Cheeger's finiteness theorem:** Given constants C, D and V, there are only finitely many (up to diffeomorphism) compact n-dimensional Riemannian manifolds with sectional curvature $|K| \leq C,$ diameter $\leq$ D and volume $\geq$ V.

3. **Gromov's almost flat manifolds.** There is an $\varepsilon_n > 0$ such that if an n-dimensional Riemannian manifold has a metric with sectional curvature $|K| \leq \varepsilon_n$ and diameter ≤ 1 then its finite cover is diffeomorphic to a nil manifold.

1.13 Sectional curvature bounded below

1. **Cheeger-Gromoll's Soul theorem**:If M is a non-compact complete non-negatively curved n-dimensional Riemannian manifold, then M contains a compact, totally geodesic sub manifold S such that M is diffeomorphic to the normal bundle of S (S is called the soul of M.) In particular, if M has strictly positive curvature everywhere, then it is diffeomorphic to R^n.

2. **Gromov's Betti number theorem:** There is a constant $C = C(n)$ such that if M is a compact connected n-dimensional Riemannian manifold with positive sectional curvature then the sum of its Betti numbers is at most C.

3. **Grove–Petersen's finiteness theorem**. Given constants C, D and V, there are only finitely many homotopy types of compact n-dimensional Riemannian manifolds with sectional curvature K ≥ C, diameter ≤ D and volume ≥ V.

1.14 Sectional curvature bounded above

1. The Cartan–Hadamard theorem states that a complete simply connected Riemannian manifold M with non positive sectional curvature is diffeomorphic to the Euclidean space R^n with n = dim M via the exponential map at any point. It implies that any two points of a simply connected complete Riemannian manifold with non positive sectional curvature are joined by a unique geodesic.
2. The geodesic flow of any compact Riemannian manifold with negative sectional curvature is ergodic.
3. If M is a complete Riemannian manifold with sectional curvature bounded above by a strictly negative constant k then it is a CAT (k) space. Consequently, its fundamental group $\Gamma = \pi_1(M)$ is Gromov hyperbolic. This has many implications for the structure of the fundamental group:

- it is finitely presented;
- the word problem for Γ has a positive solution;
- the group Γ has finite virtual cohomological dimension;
- it contains only finitely many conjugacy classes of elements of finite order;
- the abelian subgroups of Γ are virtually cyclic, so that it does not contain a subgroup isomorphic to Z×Z.

1.15 Ricci curvature bounded below

1. Myers theorem. If a compact Riemannian manifold has positive Ricci curvature then its fundamental group is finite.
2. Splitting theorem. If a complete n-dimensional Riemannian manifold has nonnegative Ricci curvature and a straight line (i.e. a geodesic which minimizes distance on each interval) then it is isometric to a direct product of the real line and a complete (n-1)-dimensional Riemannian manifold which has nonnegative Ricci curvature.
3. Bishop–Gromov inequality. The volume of a metric ball of radius r in a complete n-dimensional Riemannian manifold with positive Ricci curvature has volume at most that of the volume of a ball of the same radius r in Euclidean space.
4. Gromov's compactness theorem. The set of all Riemannian manifolds with positive Ricci curvature and diameter at most D is pre-compact in the Gromov-Hausdorff metric.

1.16 Negative Ricci curvature

1. The isometry group of a compact Riemannian manifold with negative Ricci curvature is discrete.
2. Any smooth manifold of dimension n ≥ 3 admits a Riemannian metric with negative Ricci curvature.[3] (This is not true for surfaces.)

1.17 Positive scalar curvature

1. The n-dimensional torus does not admit a metric with positive scalar curvature.
2. If the injectivity radius of a compact n-dimensional Riemannian manifold is ≥ π then the average scalar curvature is at most n (n-1).

1.18 Gauss intrinsic invariant theorem

One of the other extrinsic numerical invariants of a surface is the mean curvature K_m defined as the *sum* of the principal curvatures. It is given by the formula $K_m = \frac{eG+gE-2fF}{EG\_-F^2}$. The coefficients of the first and second fundamental forms satisfy certain compatibility conditions known as the Gauss-Codazzi equations; they involve the Christoffel symbols $\Gamma_{i,j}^{k}$ associated with the first fundamental form: $e_y - f_x = e\,\Gamma_{12}^{1} + f(\Gamma_{12}^{2} - \Gamma_{11}^{1}) - g\Gamma_{11}^{2}$

And $f_y - g_x = e\,\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\,\Gamma_{12}^2$. These equations can also be succinctly expressed and derived in the language of connection forms due to Élie Cartan. Pierre Bonnet proved that two quadratic forms satisfying the Gauss-Codazzi equations always uniquely determine an embedded surface locally. For this reason the Gauss-Codazzi equations are often called the fundamental equations for embedded surfaces, precisely identifying where the intrinsic and extrinsic curvatures come from. They admit generalizations to surfaces embedded in more general Riemannian manifolds.

VI. Differential Geometry Of Surfaces

Surface with various additional structures, most often, a Riemannian metric. Surfaces have been extensively studied from various perspectives:

Extrinsically: Relating to their embedding in Euclidean space

Intrinsically: Reflecting their properties determined solely by the distance within the surface as measured along curves on the surface.

Carl Friedrich Gauss (1825-1827) showed that curvature was an intrinsic property of a surface, independent of its isometric embedding in Euclidean space. Surfaces naturally arise as graphs of functions of a pair of variables, and sometimes appear in parametric form or as loci associated to space curves. Lie groups can be used to describe surfaces of constant Gaussian curvature; they also provide an essential ingredient in the modern approach to intrinsic differential geometry through connections. This is well illustrated by the non-linear Euler-Lagrange equations in the calculus of variations: although Euler developed the one variable equations to understand geodesics, defined independently of an embedding, one of Lagrange's main applications of the two variable equations was to minimal surfaces, a concept that can only be defined in terms of an embedding.

1.19 Determination of paths of shortest length between two fixed points on the surface

The Gaussian curvature at a point on an embedded smooth surface given locally by the equation $z = F(x, y)$ In E^3, is defined to be the product of the principal curvatures at the point; the mean curvature is defined to be their average. The principal curvatures are the maximum and minimum curvatures of the plane curves obtained by intersecting the surface with planes normal to the tangent plane at the point. If the point is (0, 0, 0) with tangent plane *z* = 0, then, after a rotation about the *z*-axis setting the coefficient on *xy* to zero, *F* will have the Taylor series expansion $\frac{1}{2}K_1x^2 + \frac{1}{2}K_2y^2 + \cdots$.

The principal curvatures are k_1 and k_2 in this case, the Gaussian curvature is given by $K = K_1 * K_2$ and the mean curvature by $K_m = \frac{1}{2}(K_1 + K_2)$ since K and K_m are invariant under isometrics of $\mathbf{E}^3$, in general $K = \frac{RT - S^2}{(1+P^2+Q^2)^2}$ and where the derivatives at the point are given by $P = F_x,\ Q = F_y,\ R = F_{x,x},\ S = F_{x,y},\ and\ T = F_{y,y}$. For every oriented embedded surface the Gauss map is the map into the unit sphere sending each point to the (outward pointing) unit normal vector to the oriented tangent plane at the point. In coordinates the map sends (x, y, z) to $N(x, y, z) = \frac{1}{\sqrt{1+p^2+q^2}}(P, Q, -1)$. Direct computation shows that: *the Gaussian curvature is the Jacobian of the Gauss map*.

Example 6.1: The surface of revolution obtained by rotating the curve $x = 2 + \cos z$ about the *z*-axis

6.2 Surfaces of revolution

A surface of revolution can be obtained by rotating a curve in the xz plane about the *z*-axis, assuming the curve does not intersect the *z*-axis. Suppose that the curve is given by$x = \varphi(t), z = \varphi(t)$, with *t* lies in (*a*, *b*), and is parameterized by arc length, so that $\vartheta^2 + \varphi^2 = 1$.Then the surface of revolution is the point set

$M = \{(\varphi(t)cos\theta, \vartheta(t)sin\theta, \varphi(t)): t\epsilon(a,b), \theta\epsilon[0,2\pi)\}$.The Gaussian curvature and mean curvature are given by$K = -\frac{\vartheta}{\vartheta}, K_m = \frac{-\varphi(\varphi\emptyset - \varphi''\emptyset')}{2\emptyset}$

6.3 Quadric surfaces

Consider the quadric surface defined by$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$, this surface admits a parameterization$x = \sqrt{\frac{a(a-u)(a-v)}{(a-b)(a-c)}}, y = \sqrt{\frac{b(b-u)(b-v)}{(b-a)(b-c)}}\ z = \sqrt{\frac{c(c-u)(c-v)}{(c-b)(c-a)}}$. The Gaussian curvature and mean curvature are given by$K = \frac{abc}{u^2v^2}$: $k_m = -(u+v)\sqrt{[\frac{abc}{u^3v^3}]}$

6.4 Ruled surfaces

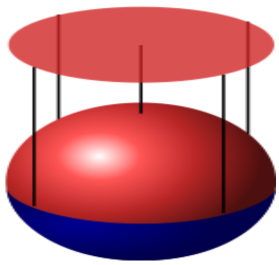
A ruled surface is one which can be generated by the motion of a straight line in $\mathbf{E}^3$. Choosing a *directrix* on the surface, i.e. a smooth unit speed curve $c(t)$ orthogonal to the straight lines, and then choosing $u(t)$ to be unit vectors along the curve in the direction of the lines, the velocity vector $v = c_t$ and u satisfy $u.v = 0, \|u\| = 1; \|v\| = 1.$ The surface consists of points$c(t) + s.u(t)$ as s and t vary. Then, if $a = \|u_t\|; b = u_t.v;\ \propto = -\frac{b}{a^2};\ \beta = \frac{(a^2-b^2)}{a^2}$ The Gaussian curvature of the ruled surface vanishes if and only if u_t and v are proportional, This condition is equivalent to the surface being the envelope of the planes along the curve containing the tangent vector v and the orthogonal vector u, i.e. to the surface being developable along the curve. More generally a surface in $\mathbf{E}^3$ has vanishing Gaussian curvature near a point if and only if it is developable near that point. (An equivalent condition is given below in terms of the metric.)

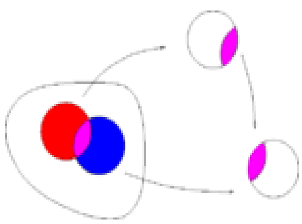
Remarks:

- The unit sphere in $\mathbf{E}^3$ has constant Gaussian curvature +1.
- The Euclidean plane and the cylinder both have constant Gaussian curvature 0.
- The surfaces of revolution with $\varphi_{tt} = \varphi$ have constant Gaussian curvature –1.

6.5 Local metric structure

A chart for the upper hemisphere of the 2-sphere obtained by projecting onto the *x*-*y*-plane. Coordinate changes between different local charts must be smooth For any surface embedded in Euclidean space of dimension 3 or higher, it is possible to measure the length of a curve on the surface, the angle between two curves and the area of a region on the surface. This structure is encoded infinitesimally in a Riemannian metric on the surface through *line elements* and *area elements*.





Classically in the nineteenth and early twentieth century's only surfaces embedded in R^3 were considered and the metric was given as a 2×2 positive definite matrix varying smoothly from point to point in a local parameterization of the surface. The idea of local parameterization and change of coordinate was later formalized through the current abstract notion of a manifold, a topological space where the smooth structure is given by local charts on the manifold, exactly as the planet Earth is mapped by atlases today. Changes of coordinates between different charts of the same region are required to be smooth. Just as contour lines on real-life maps encode changes in elevation, taking into account local distortions of the Earth's surface to calculate true distances, so the Riemannian metric describes distances and areas "in the small" in each local chart. In each local chart a Riemannian metric is given by smoothly assigning a 2×2 positive definite matrix to each point; when a different chart is taken, the matrix is transformed according to the Jacobian matrix of the coordinate change. The manifold then has the structure of a 2-dimensional Riemannian manifold.

6.6 Line and area elements

Taking a local chart, for example by projecting onto the *x*-*y* plane (*z* = 0), the line element *ds* and the area element *dA* can be written in terms of local coordinates as $e\,dx^2 + 2f\,dx\,dy + g\,dy^2$ and

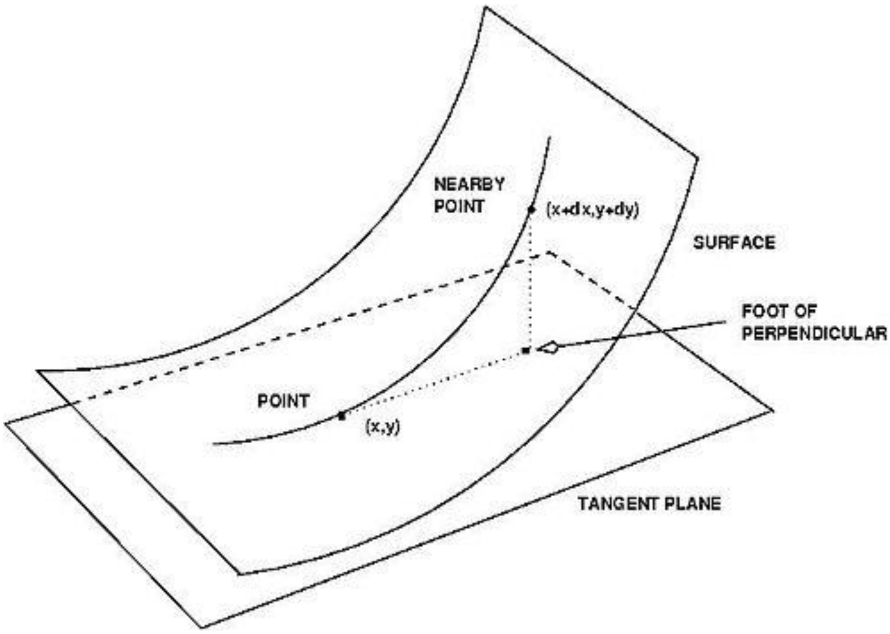
$\boldsymbol{dA = (EG - F^2)dx\,dy}$. the expression $\boldsymbol{E\,dx^2 + 2F\,dx\,dy + G\,dy^2}$ is called the first fundamental form**.**

The matrix $\begin{bmatrix} E(x,y) & F(x,y) \\ F(x,y) & G(x,y) \end{bmatrix}$ is required to be positive-definite and to depend smoothly on *x* and *y*. In a similar way line and area elements can be associated to any abstract Riemannian 2-manifold in a local chart.

6.7 Second fundamental form

Take a point (*x*, *y*) on the surface in a local chart. The Euclidean distance from a nearby point $(x + dx,\ y + dy)$ to the tangent plane at (*x*, *y*), i.e. the length of the perpendicular dropped from the nearby point to the tangent plane, has the form $e\,dx^2 + 2f\,dx\,dy + g\,dy^2$ plus third and higher order corrections.

The above expression, a symmetric bilinear form at each point, is the second fundamental form. It is described by a 2 × 2 symmetric matrix $\begin{bmatrix} e(x,y) & f(x,y) \\ f(x,y) & g(x,y) \end{bmatrix}$ which depends smoothly on *x* and *y*. The Gaussian curvature can be calculated as the ratio of the determinants of the second and first fundamental forms: $K = \frac{eg - f^2}{EG - F^2}$



VII. Conclusion

Differential geometry is normally considered as a speculation of the Riemannian geometry. The historical backdrop of improvement of Finsler geometry can be partitioned into four periods. The primary period of the historical backdrop of advancement of Finster geometry started in 1924, when three geometricians J.H. Taylor, J.L. Synge and L. Berwald at the same time began work in this field. Berwald is the main man who has presented the idea of association in the hypothesis of Finsler spaces. He is the maker of Finsler geometry and, besides, the author. He has built up a hypothesis with specific reference to the hypothesis of bend in which the

Ricci lesa does not hold great. J.H. taylor gave the name 'Finsler space' to the complex outfitted with this summed up metric. The second time frame started in 1934, when E. Cartan distributed his proposition on Finsler geometry. He demonstrated that it was to be sure conceivable to characterize association coefficients and subsequently covariant subordinates with the end goal that the Ricci lemma is fulfilled. On this premise Cartan built up the hypothesis of curvature tensor and torsion. Every single consequent examination considering the geometry of Finsler spaces were ruled by this approach. A few mathematicians, for example, E.T. Davies , Golab , H. Hombu , O. Varga , V.V. Wagner have considered Finsler g•ometry along Cartan's approach. They have communicated the conclusion that the hypothesis has achieved its last shape. This hypothesis makes certain gadgets, which fundamentally includes the thought of a space whose components are not the purposes of the basic complex, but rather the line-components of the last mentioned, which shapes a (2n-1) — dimensional assortment. This encourages what Cartan called 'Euclidean association' which by method for specific proposes might be gotten extraordinarily from the crucial metric capacity.

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