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A note on norm attaining operators

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***Abstract**: We show the norm attaining quadraicallyhyponormal weighted shift is subnormal . Also , We show that there is a Banach space X such that the set of norm attaining operators from Xto any infinite dimensional space $L_1(\mu)$ is not dense .*

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I. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebraof bounded linear operators on $\mathcal{H}$. An operator $A \in \mathcal{L}(H)$ is said tobe normal if $A^*A = AA^*$hyponormal if $A^*A \geq AA^*$ and subnormalif $A = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $K \supseteq \mathcal{H}$. Anoperator $A \in \mathcal{L}(\mathcal{H})$ is said to be paranormal if $\|A^2x\| \geq \|Ax\|^2$for allunit vector $x \in H$. An operator $A \in \mathcal{L}(\mathcal{H})$ is called norm attaining if there is an $x \in \mathcal{H}$ with $\|x\| = 1$ and $\|Ax\| = \|A\|$.

The Bishop -Phelps theorem, the origin of the so-called ‘‘perturbedoptimization principles,’’ asserts that the set of norm-attaining functionalson a Banach space is norm dense in the set of all bounded functionals . Given Banach spaces$X$and Y, let us consider the Banach space $L(X, Y)$. of bounded linearoperators from X into $Y$and let us denote by $NA(X, Y)$. the set ofnorm-attaining operators; that is, $A \in NA(X, Y)$. if for some element $x$in the unit sphere X. such that $\|Ax\| = \|A\|$. In mentioned paper by E.Bishop and R.Phelpsauther raised the problem if $NA(X, Y)$. is norm dense in $L(X, Y)$.

II. Results

We start from a basic criterion for norm attaining operators:

Lemma 1. If $A \in \mathcal{L}(\mathcal{H})$is a norm attaining operator if and only if$\|A\|^2 \in \sigma_p(A^*A)$; where $\sigma_p(S)$ denote the point spectrum of $S \in \mathcal{L}(\mathcal{H})$.

Proof. Observe that $\|Ax\| = \|A\|\|x\|$ if and only if $\langle (A^*A - \|A\|^2)x, x\rangle = 0$Since $A^*A - \|A\|^2$is hermitian, we can see that $\langle (A^*A - \|A\|^2)x, x\rangle = 0$ if and only if $A^*Ax - \|A\|^2x$ or equivalently, $x \in Ker(A^*A - \|A\|^2I)$. Thus A is a norm attaining operator if and only if $\|A\|^2 \in \sigma_p(A^*A)$.

Let $\{\beta_n\}_{n=0}^{\infty}$ be a bounded sequence of positive real numbers, and let$A_\beta: \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by$A_\beta g_n = \beta_n g_{n+1}$ (all $n \geq 0$), where $\{g_n\}_{n=0}^{\infty}$is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$.(where $\mathbb{Z}_+$ is the set of non-negative integers) . It is well-known that $A_\beta$is hyponormal if and only if $\beta_n < \beta_{n+1}$for all $n \geq 0$.

Theorem 2. $A_\beta$is norm attaining if and only if $\|A_\beta\| = \beta_i$ for some$i$.

Proof. Since$A_\beta^* A_\beta = diag\{\beta_0^2, \beta_1^2, \ldots.\}$, we have $\sigma_p(A_\beta^* A_\beta) = \{\beta_0^2, \beta_1^2, \ldots.\}$. The desired result now follows from Lemma 1.

In addition to its usefulness to produce examples of hyponormal weighted shifts T for which $A_\beta + \lambda A_\beta^2$ is not hyponormal (for some complex number λ),
For, if $\beta_0 < \beta_1 < \beta_2 = \beta_3 \ldots.$, one knows that the associated A_β can't be subnormal,so one could use the freedom inβ_0 and $\beta_1$to build such an example. However, such anattempt is doomed to fail, as the following theorem shows. First, we need a definition.

Definition 3: Let A_β be a Hilbert space operator. We call A_β quadraticall hyponormal if $A_\beta + \lambda A_\beta^2$ is hyponormal for every complex number λ.

Theorem4:. Let A_β be a subnormal weightedshift with weight sequence $\{\beta_n\}_{n=0}^{\infty}$if $\beta_n = \beta_{n+1} = \cdots$for some$n \geq 0$then

$$\beta_1 = \beta_2 = \beta_3 = \cdots$$

Corollary 5:. Let A_β be a norm attaining hyponormal weightedshift. Then $\beta_n = \beta_{n+1}$ =…for some $n \geq 0$.

Proof. By Theorem 4, we have that $\|A_\beta\| = \beta_n = \max_i \beta_i$ for some $n \geq 0$. But since A_β is hyponormal, the corresponding weight sequence is monotonically increasing. Thus, $\beta_n = \beta_{n+1} = \cdots$

Theorem 6: If A_β is 2-hyponormal and$\beta_n = \beta_{n+1}$ for some n, then $\beta_1 = \beta_2 = \beta_3 = \cdots$, A_β is subnormal.

Although the norm attaining operators are dense in $\mathcal{L}(\mathcal{H})$, we can-not expect that every hyponormaloperator is a normattaining operator.

Corollary7: Let $\beta \equiv \{\beta_n\}_{n=0}^{\infty}$be a strictly increasing bounded sequence. Then A_β is hyponormal (and hence paranormal), but not normattaining.

III. Lorentz Spaces

Let us start by recalling the definition of Lorentz sequence spaces and preduals a family of classical Banach spaces .

By *admissible sequence* w we shall mean a decreasing sequence $w = (w(n)$ of positive numbers such that $w(1) = 1$and $w \in c_0 \setminus \ell_1$,the Banach space of all sequences of scalars $b = (b(n)$ for which

$$\|b\| = \sup_{\pi} \left(\sum_{j=1}^{\infty} |b(\pi(n)|^p \ \ w(n) \right)^{1/p},$$

where πranges over all permutations of the integers ,denote by $d_*(w,p)$ is called Lorentz sequence if $p = 1$ it is known [8,15]that $d(w,1)$has predual$d_*(w,1)$

which is defined by

$$d_*(w) = \left\{ b \in c_0 : \lim_{n \to \infty} \frac{\sum_{j=1}^{n} b^*(j)}{\sum_{j=1}^{n} w(j)} = 0 \right\},$$

where b^* is the decreasing rearrangement of $\{|b(n)|\}$ The norm of

$d_*(w)$ is given by

$$\|b\| = \sup_n \frac{\sum_{j=1}^{n} b^*(j)}{\sum_{j=1}^{n} w(j)}$$

Thus $\|b\| \leq 1$ if and only if

$$\sum_{k \in J} |b(k)| \leq \sum_{j=1}^{n} w(j)$$

For any $n \in N$ and any set $J \subset N$ with n elements

$d_*(w)$predual of the Lorentz sequencespace $d(w,1)$,and $d_*(w)$ is space with symmetric basis $\{e_n\}$ that shares some the properties of c_0. Also the space $d_*\left(\frac{1}{n}\right)$ was used by [GO] to get an example of a space X such that the closure of the $NA(X,\ell_p)$is the set of compact operators for any $1 < p < \infty$ and so the set of norm attaining operators is not dense , since the space $d_*\left(\frac{1}{n}\right)$ is a subset of ℓ_p.

We do have :

Theorem 8:*Let* $w \notin \ell_1$ be a decreasing sequence of positive real numbers and μ

any positive measure. The following assertions hold:

i) $\overline{NA(d_*(w,1)L_1(\mu))} = K(d_*(w,1)L_1(\mu))$

ii) If μ is purely atomic, the set of norm attaining operators from $d_*(w,1)$to$L_1(\mu)$is dense.

iii) If μ is not purely atomic and σ-finite, then

$$\overline{NA(d_*(w,1)L_1(\mu))} = L(d_*(w,1)L_1(\mu)) \Leftrightarrow w \notin \ell_1$$

Theorem 9: Assume that $w \in \ell_2 \backslash \ell_1$. For the complex Lorentz sequence space $d(w,1)$ and its canonical predual$d_*(w,1)$, then$NA(d_*(w,1), d(w,1))$ is not dense in $L(d_*(w,1), d(w,1))$.

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