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Stability Of Non- Additive Functional Equation

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Abstract : *In this paper, the generalized n-dimensional quadratic functional equation of the form*

$$\sum_{\substack{i=1\\ i\neq j\neq k}}^{n} f\left(-x_i - x_j - x_k + \sum_{\substack{i=1\\ i\neq j\neq k}}^{n} x_i\right) = \left(\frac{n^3-15n^2+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n} f\left(x_i+x_j\right) + \left(\frac{-n^4+17n^3-80n^2+136n-72}{6}\right)\sum_{i=1}^{n} f\left(x_i\right)$$

when n is a positive integer with $\square -\{0,1,2,3\}$ *is introduce in Fuzzy Normed Space. Further, the general solution is obtained. The stability of the general solution obtained is verified by the generalized Hyers-Ulam method associated with direct and fixed point methods.*
Keywords *- Banach Sapce, Fixed Point, Fuzzy Normed Space, Hyers-Ulam Stability, Quadratic Functional Equation.*

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I. Introduction

The quadratic functional equation was first introduced by J. M. Rassias, who solved Ulam stability. The quadratic function $f(x)=cx^2$ satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

which is called quadratic functional equation and it was investigated by leading experts F. Skof [28], P. W. Cholewa [6], S. Czerwik [7] and J. M. Rassias [25].
The solution and stability of the subsequent quadratic functional equations

$$f\left(\sum_{i=1}^{n} x_i\right)+\sum_{1\le i\le j\le n} f\left(x_i-x_j\right)=n\sum_{i=1}^{n} f\left(x_i\right) \tag{1.2}$$

$$\sum_{1\le i\le j\le n}\left(f\left(x_i+x_j\right)+f\left(x_i-x_j\right)\right)=2(n-1)\sum_{i=1}^{n} f\left(x_i\right) \tag{1.3}$$

$$\begin{aligned} f\left(nx+n^2y+n^3z\right)+f\left(nx-n^2y+n^3z\right)+f\left(nx+n^2y-n^3z\right)+f\left(-nx+n^2y+n^3z\right) \\ =n\left[f(x)-f(-x)\right]+n^2\left[f(y)-f(-y)\right]+n^3\left[f(z)-f(-z)\right] \\ +2n^2\left[f(x)+f(-x)\right]+2n^4\left[f(y)+f(-y)\right]+2n^6\left[f(z)+f(-z)\right] \end{aligned} \tag{1.4}$$

were discussed by J. H. Bae [1], T. Eungrasamee et al., [9] and S. Murthy et al., [19].
In this paper, the authors introduce a new type of n-dimensional quadratic functional equation

$$\sum_{\substack{i=1\\ i\neq j\neq k}}^{n} f\left(-x_i - x_j - x_k + \sum_{\substack{i=1\\ i\neq j\neq k}}^{n} x_i\right) = \left(\frac{n^3-15n^2+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n} f\left(x_i+x_j\right) + \left(\frac{-n^4+17n^3-80n^2+136n-72}{6}\right)\sum_{i=1}^{n} f\left(x_i\right) \tag{1.5}$$

where n is a positive integer with $\square -\{0,1,2,3\}$.

Theorem A. (Banach's contraction principle): Let (X, d) be a complete metric space and consider a mapping $T: X \to X$ which is strictly contractive mapping, that is
(A1) $d(Tx,Ty)\le Ld(x,y)$ for some (Lipchitz constant) L<1, then

i) The mapping T has one and only fixed point $x^* = T\left(x^*\right)$;

ii) The fixed point for each given element x^* is globally attractive that is

(A2) $\lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;

iii) One has the following estimation inequalities:

(A3) $d\left(T^n x, x^*\right) \le \frac{1}{1-L} d\left(T^n x, T^{n+1} x\right)$, for all $n \ge 0, \quad x \in X$.

(A4) $d\left(x, x^*\right) \le \frac{1}{1-L} d\left(x, x^*\right), \quad \forall \ x \in X$.

Theorem B. (The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to Y$ with Lipschitz constant L. Then, for each given element $x \in X$, either

(B1) $d\left(T^n x, T^{n+1} x\right) = \infty, \quad \forall \ n \ge 0$ or

(B2) there exists natural number n_0 such that:

i) $d\left(T^n x, T^{n+1} x\right) < \infty$ for all $n \ge n_0$.

ii) The sequence $\left(T^n x\right)$ is convergent to a fixed point y^* of T

iii) y^* is the unique fixed point of T in the set $Y = \left\{y \in X : d\left(T^n x, y\right) < \infty\right\}$;

iv) $d\left(y^*, y\right) \le \frac{1}{1-L} d\left(y, Ty\right)$ for all $y \in L$.

II. General Solution of the Functional Equation (1.5)

In this segment, the author obtains the general solution of the functional equation (1.5). All over this segment, let X and Y be real vector space.

Theorem 2.1 Let X and Y be a real vector spaces. The mapping $f : X \to Y$ satisfies the functional equation (1.5) for all $x_1, x_2, ..., x_n \in X$, then $f : X \to Y$ satisfies the functional equation (1.1) for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies the functional equation (1.5). Letting $\left(x_1, x_2, x_3, ..., x_n\right)$ by $\left(0, 0, 0, ..., 0\right)$ in (1.5), we get

$$f(0) = 0,$$

Replacing $\left(x_1, x_2, x_3, ..., x_n\right)$ by $\left(x, 0, 0, ..., 0\right)$ in (1.5), we obtain

$$f\left(-x\right) = f\left(x\right)$$

for all $x \in X$. Hence f is an even function. Replacing $\left(x_1, x_2, x_3, ..., x_n\right)$ by $\left(x, x, 0, ..., 0\right)$ and using evenness in (1.5), we receive

$$f\left(2x\right) = 2^2 f\left(x\right)$$

for all $x \in X$. Setting $\left(x_1, x_2, x_3, ..., x_n\right)$ by $\left(x, x, x, 0, ..., 0\right)$ in (1.5), we have

$$f\left(3x\right) = 3^2 f\left(x\right).$$

In general, for any positive integer a, we get

$$f\left(ax\right) = a^2 f\left(x\right)$$

for all $x \in X$. Now substituting $\left(x_1, x_2, x_3, ..., x_n\right)$ by $\left(x, y, 0, ..., 0\right)$ in (1.5), we reach (1.1) as preferred.

In section 3 and 4, we take X be a normed space and Y be a Banach Space. For notational handiness, we define a function $Q: X \to Y$ by

$$Q(x_1, x_2, ..., x_n) = \sum_{\substack{i=1 \\ i\neq j\neq k}}^{n} f\left(-x_i - x_j - x_k + \sum_{\substack{i=1 \\ i\neq j\neq k}}^{n} x_i\right) - \left(\frac{n^3 - 15n^2 + 62n - 72}{6}\right)\sum_{\substack{i=1 \\ i\neq j}}^{n} f(x_i + x_j) - \left(\frac{-n^4 + 17n^3 - 80n^2 + 136n - 72}{6}\right)\sum_{i=1}^{n} f(x_i)$$

for all $x_1, x_2, ..., x_n \in X$.

III. Stability Results for (1.5): Direct Method

In this segment, we prove the generalized Ulam-Hyers stability of the n-dimensional functional equation (1.5) in Banach space with the help of direct method.
In this segment, authors consider X to be a real vector space and Y be a Banach Space.

Theorem 3.1 Let $j \in \{-1,1\}$. Let $\chi : X^n \to [0, \infty)$ be a function such that $\sum_{k=0}^{\infty} \frac{\chi(2^{kj} x_1, 2^{kj} x_2, ..., 2^{kj} x_n)}{2^{2kj}}$ converges in $\mathbb{R}$ and $\lim_{k\to\infty} \frac{\chi(2^{kj} x_1, 2^{kj} x_2, ..., 2^{kj} x_n)}{2^{2kj}} = 0$ (3.1)

for all $x_1, x_2, ..., x_n \in X$. Let $f : X \to Y$ be an even function satisfying the inequality

$$\| Q(x_1, x_2, ..., x_n) \| \le \chi(x_1, x_2, ..., x_n) \tag{3.2}$$

for all $x_1, x_2, ..., x_n \in X$. Then there exists a unique quadratic mapping $G: X \to Y$ which satisfies the functional equation (1.5) and

$$\| f(x) - G(x) \| \le \frac{1}{4(n^2 - 5n + 6)} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\chi(2^{kj} x, 2^{kj} x, 0, ..., 0)}{2^{2kj}} \tag{3.3}$$

for all $x \in X$. The mapping $G(x)$ is defined by

$$G(x) = \lim_{k\to\infty} \frac{f(2^{kj} x)}{2^{2kj}} \tag{3.4}$$

for all $x \in X$.

Proof. Assume that $j = 1$. Replacing $(x_1, x_2, ..., x_n)$ by $(x, x, 0, ..., 0)$ in (3.2), we get

$$\left\| (n^2 - 5n + 6) f(2x) - 4(n^2 - 5n + 6) f(x) \right\| \le \chi(x, x, 0, ..., 0) \tag{3.5}$$

for all $x \in X$. It follows from (3.5), we arrive

$$\left\| \frac{f(2x)}{2^2} - f(x) \right\| \le \frac{\chi(x, x, 0, ..., 0)}{4(n^2 - 5n + 6)} \tag{3.6}$$

Replacing x by $2x$ in (3.6), we have

$$\left\| \frac{f(2^2 x)}{2^2} - f(2x) \right\| \le \frac{\chi(2x, 2x, 0, ..., 0)}{4(n^2 - 5n + 6)} \tag{3.7}$$

for all $x \in X$. It follows from (3.7), we get

$$\left\| \frac{f(2^2 x)}{2^4} - \frac{f(2x)}{2^2} \right\| \le \frac{1}{2^2} \frac{\chi(2x, 2x, 0, ..., 0)}{4(n^2 - 5n + 6)} \tag{3.8}$$

for all $x \in X$. Adding (3.6) and (3.8), we receive

$$\left\|\frac{f\left(2^2 x\right)}{2^4}-f\left(x\right)\right\| \le \frac{1}{4\left(n^2-5n+6\right)}\left[\chi\left(x,x,0,...,0\right)+\frac{\chi\left(2x,2x,0,...,0\right)}{2^2}\right] \tag{3.9}$$

for all $x \in X$. It follows from using (3.6), (3.8) and (3.9), generalizing we receive

$$\left\|\frac{f\left(2^n x\right)}{2^{2n}}-f\left(x\right)\right\| \le \frac{1}{4\left(n^2-5n+6\right)}\sum_{k=0}^{n-1}\frac{\chi\left(2^k x,2^k x,0,...,0\right)}{2^{2k}}$$

$$\le \frac{1}{4\left(n^2-5n+6\right)}\sum_{k=0}^{\infty}\frac{\chi\left(2^k x,2^k x,0,...,0\right)}{2^{2k}} \tag{3.10}$$

for all $x \in X$. In order to prove convergence of the sequence $\left\{\frac{f\left(2^k x\right)}{2^{2k}}\right\}$, replace x by $2^l x$ and dividing 2^{2l} in (3.10), for any $k,l>0$ to deduce

$$\left\|\frac{f\left(2^{k+l} x\right)}{2^{2(k+l)}}-\frac{f\left(2^l x\right)}{2^{2l}}\right\| = \frac{1}{2^{2l}}\left\|\frac{f\left(2^{k+l} x\right)}{2^{2k}}-f\left(2^l x\right)\right\|$$

$$\le \frac{1}{4\left(n^2-5n+6\right)}\sum_{k=0}^{n-1}\frac{\chi\left(2^{k+l} x,2^{k+l} x,0,...,0\right)}{2^{2(k+l)}}$$

$$\le \frac{1}{4\left(n^2-5n+6\right)}\sum_{k=0}^{\infty}\frac{\chi\left(2^{k+l} x,2^{k+l} x,0,...,0\right)}{2^{2(k+l)}} \tag{3.11}$$

$$\to 0 \quad as \quad l \to \infty$$

for all $x \in X$. Hence the sequence $\left\{\frac{f\left(2^k x\right)}{2^{2k}}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $G: X \to Y$ such that

$$G\left(x\right)=\lim_{k\to\infty}\frac{f\left(2^k x\right)}{2^{2k}}$$

for all $x \in X$. Letting $k \to \infty$ in (3.10) we get the result (3.3) holds for all $x \in X$. To prove that G satisfies (1.5), replacing $\left(x_1,x_2,...,x_n\right)$ by $\left(2^k x_1,2^k x_2,...,2^k x_n\right)$ and dividing 2^{2k} in (3.2), we have

$$\frac{1}{2^{2k}}\left\|Q\left(2^k x_1,2^k x_2,...,2^k x_n\right)\right\| \le \frac{1}{2^{2k}}\chi\left(2^k x_1,2^k x_2,...,2^k x_n\right)$$

for all $x_1,x_2,...,x_n \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $G\left(x\right)$, we see that

$$G\left(x_1,x_2,...,x_n\right)=0$$

for all $x_1,x_2,...,x_n \in X$. Hence G satisfies (1.5). To show that G is unique. Let $H\left(x\right)$ be an another quadratic mapping satisfying (1.5) and (3.3), then

$$\left\|G\left(x\right)-H\left(x\right)\right\| \le \frac{1}{2^{2l}}\left\|G\left(2^{2l} x\right)-f\left(2^{2l} x\right)\right\|+\left\|f\left(2^{2l} x\right)-H\left(2^{2l} x\right)\right\|$$

$$\le \frac{1}{4\left(n^2-5n+6\right)}\sum_{k=0}^{\infty}\frac{\chi\left(2^{k+l} x,2^{k+l} x,0,...,0\right)}{2^{2(k+l)}}$$

$$\to 0 \quad as \quad l \to \infty$$

for all $x \in X$. Hence G is unique. Now replacing x by $\frac{x}{2}$ in (3.5) we have

$$\left\|\left(n^2-5n+6\right)f\left(x\right)-4\left(n^2-5n+6\right)f\left(\frac{x}{2}\right)\right\| \le \chi\left(\frac{x}{2},\frac{x}{2},0,...,0\right) \tag{3.12}$$

for all $x \in X$. It follows from (3.12), we get

$$\left\|f\left(x\right)-4f\left(\frac{x}{2}\right)\right\| \le \frac{1}{\left(n^2-5n+6\right)}\chi\left(\frac{x}{2},\frac{x}{2},0,...,0\right) \tag{3.13}$$

for all $x \in X$. The rest of the proof is similar to that $j=1$. Hence for $j=-1$, also the theorem is true. This completes the proof of the Theorem.

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.5).

Corollary 3.2 Let ε and s be non-negative real numbers. If a function $f : X \to Y$ satisfying the inequality

$$\left\|Q\left(x_1,x_2,...,x_n\right)\right\| \le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{n}\|x_i\|^s\right\}, \\ \varepsilon\left\{\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right\}, \end{cases} \tag{3.14}$$

for all $x_1,x_2,...,x_n \in X$. Then there exists a unique quadratic function $G : X \to Y$ such that

$$\left\|f\left(x\right)-G\left(x\right)\right\| \le \begin{cases} \frac{\varepsilon}{3\left(n^2-5n+6\right)} \\ \frac{2\varepsilon\|x\|^s}{\left(n^2-5n+6\right)\left(2^2-2^s\right)} & ; s \neq 2 \\ \frac{2\varepsilon\|x\|^{ns}}{\left(n^2-5n+6\right)\left(2^2-2^{ns}\right)} & ; s \neq \frac{2}{n} \end{cases} \tag{3.15}$$

for all $x \in X$.

Proof. If we replace,

$$\chi\left(x_1,x_2,...,x_n\right) \le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{n}\|x_i\|^s\right\}, \\ \varepsilon\left\{\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right\}, \end{cases}$$

in Theorem 3.1, we get (3.15).

IV. Stability Results for (1.5): Fixed Point Method

In this segment, we prove the generalized Ulam-Hyers stability of the n-dimensional functional equation (1.5) in Banach space with the help of the fixed point method.

Theorem 4.1 Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^n \to [0,\infty)$ with the condition

$$\lim_{k\to\infty}\frac{\chi\left(\psi_i^k x_1,\psi_i^k x_2,...,\psi_i^k x_n\right)}{\psi_i^{2k}}=0 \quad (4.1)$$

where $\psi_i=\begin{cases}2, & i=0;\\ \frac{1}{2} & i=1;\end{cases}$ satisfying the functional inequality

$$\left\|Q\left(x_1,x_2,...,x_n\right)\right\|\le\chi\left(x_1,x_2,...,x_n\right) \quad (4.2)$$

for all $x_1,x_2,...,x_n\in X$ and $n\ge 4$ if there exists $L=L(i)$ such that the function

$$x\to\rho(x)=\frac{1}{\left(n^2-5n+6\right)}\chi\left(\frac{x}{2},\frac{x}{2},0,...,0\right)$$

has the property

$$\frac{\rho(\psi_i x)}{\psi_i^2}=L\rho(x) \quad (4.3)$$

for all $x\in X$. Then there exists a unique quadratic function $G:X\to Y$ satisfying the functional equation (1.5) and

$$\left\|f(x)-G(x)\right\|\le\frac{L^{1-i}}{1-L}\rho(x) \quad (4.4)$$

for all $x\in X$.

Proof. Consider the set $\Omega=\left\{p/p:G\to H,\, p(0)=0\right\}$ and introduce the generalized metric on Ω ,

$$d(p,q)=\inf\left\{k\in(0,\infty):\left\|p(x)-q(x)\right\|\le k\rho(x),x\in G\right\}.$$

It is easy to see that (Ω,d) is complete. Define $T:\Omega\to\Omega$ by $T_p(x)=\frac{1}{\psi_i^2}p(\psi_i x)$, for all $x\in X$. For $p,q\in\Omega$ and $x\in X$, we have

$$\begin{aligned}
&d(p,q)=k\Rightarrow\left\|p(x)-q(x)\right\|\le k\rho(x),\\
&\Rightarrow\left\|\frac{p(\psi_i x)}{\psi_i^2}-\frac{q(\psi_i x)}{\psi_i^2}\right\|\le\frac{1}{\psi_i^2}k\rho(\psi_i x),\\
&\Rightarrow\left\|T_p(x)-T_q(x)\right\|\le\frac{1}{\psi_i^2}k\rho(\psi_i x),\\
&\Rightarrow\left\|T_p(x)-T_q(x)\right\|\le Lk\rho(x)\Rightarrow d\left(T_p(x),T_q(x)\right)\le kL
\end{aligned}$$

That is $d\left(T_p,T_q\right)\le Ld(p,q)$. Therefore, T is strictly contractive mapping on Ω with Lipschitz constant L. It is follows from (3.5) that

$$\left\|\left(n^2-5n+6\right)f(2x)-4\left(n^2-5n+6\right)f(x)\right\|\le\chi(x,x,0,...,0) \quad (4.5)$$

for all $x\in X$. It is follows from (4.5) that

$$\left\|\frac{f(2x)}{2^2}-f(x)\right\|\le\frac{\chi(x,x,0,...,0)}{4\left(n^2-5n+6\right)} \quad (4.6)$$

for all $x\in X$. Using (4.3) for the case $i=0$, it reduces to

$$\left\|f(x)-\frac{f(2x)}{2^2}\right\|\le\frac{1}{2^2}L\rho(x)\Rightarrow\left\|f(x)-T_p(x)\right\|\le L\rho(x)$$

for all $x\in X$. Hence, we obtain

$$d\left(T_f(x)-f(x)\right)\le L=L^{1-i}<\infty \tag{4.7}$$

for all $x\in X$. Replacing x by $\frac{x}{2}$ in (4.6), we have

$$\left\|\frac{f(x)}{2^2}-f\left(\frac{x}{2}\right)\right\|\le\frac{\chi\left(\frac{x}{2},\frac{x}{2},0,...,0\right)}{4\left(n^2-5n+6\right)} \tag{4.8}$$

for all $x\in X$. Using (4.3) for the case $i=0$, it reduce to

$$\left\|4f\left(\frac{x}{2}\right)-f(x)\right\|\le\rho(x)\Rightarrow\left\|T_f(x)-f(x)\right\|\le\rho(x)$$

for all $x\in X$. Hence, we get

$$d\left(f(x)-T_f(x)\right)\le\frac{1}{4}=L^{1-i} \tag{4.9}$$

for all $x\in X$. From (4.7) and (4.9), we can conclude

$$d\left(f(x)-T_f(x)\right)\le L^{1-i}<\infty \tag{4.10}$$

for all $x\in X$. Now from the fixed point alternative in both cases, it follows that there exists a fixed point G of T in Ω such that

$$G(x)=\lim_{k\to\infty}\frac{f\left(\psi_i^k x\right)}{\psi_i^{2k}} \tag{4.11}$$

for all $x\in X$. In order to prove $G:X\to Y$ satisfies the functional equation (1.5), the proof is similar to that of Theorem 3.1. Since G is unique fixed point of T in the set $\Delta=\left\{f\in\Omega/d\left(f,G\right)<\infty\right\}$. Therefore G is an unique function such that

$$d\left(f,G\right)\le\frac{1}{1-L}d\left(f,T_f\right)\Rightarrow d\left(f,G\right)\le\frac{L^{1-i}}{1-L}$$

$$i.e.,\left\|f(x)-G(x)\right\|\le\frac{L^{1-i}}{1-L}\rho(x)$$

for all $x\in X$. This completes the proof of the Theorem.

The following corollary is an immediate consequence of Theorem 4.1 concerning the stability of (1.5).

Corollary 4.2 Let ε and s be non-negative real numbers. If a function $f:X\to Y$ satisfies the inequality

$$\left\|Q\left(x_1,x_2,...,x_n\right)\right\|\le\begin{cases}\varepsilon,\\ \varepsilon\left\{\sum_{i=1}^{n}\left\|x_i\right\|^s\right\},\\ \varepsilon\left\{\prod_{i=1}^{n}\left\|x_i\right\|^s+\sum_{i=1}^{n}\left\|x_i\right\|^{ns}\right\},\end{cases} \tag{4.12}$$

for all $x_1,x_2,...,x_n\in X$. Then there exists an unique quadratic function such that

$$\| f(x)-G(x)\| \le \begin{cases} \dfrac{\varepsilon}{|3|\left(n^2-5n+6\right)} \\ \dfrac{2\varepsilon\|x\|^s}{\left(n^2-5n+6\right)\left|2^2-2^s\right|} & ; s\neq 2 \\ \dfrac{2\varepsilon\|x\|^{ns}}{\left(n^2-5n+6\right)\left|2^2-2^{ns}\right|} & ; s\neq \dfrac{2}{n} \end{cases} \tag{4.13}$$

for all $x \in X$.

Proof. Setting

$$\chi\left(x_1, x_2, \ldots, x_n\right) \le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{n}\|x_i\|^s\right\}, \\ \varepsilon\left\{\prod_{i=1}^{n}\|x_i\|^s + \sum_{i=1}^{n}\|x_i\|^{ns}\right\}, \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$. Now

$$\frac{\chi\left(\psi_i^k x_1, \psi_i^k x_2, \ldots, \psi_i^k x_n\right)}{\psi_i^{2k}} = \begin{cases} \dfrac{\varepsilon}{\psi_i^{2k}}, \\ \dfrac{\varepsilon}{\psi_i^{2k}}\left\{\sum_{i=1}^{n}\|\psi_i x_i\|^s\right\}, \\ \dfrac{\varepsilon}{\psi_i^{2k}}\left\{\prod_{i=1}^{n}\|\psi_i x_i\|^s + \sum_{i=1}^{n}\|\psi_i x_i\|^{ns}\right\}, \end{cases}$$

$$= \begin{cases} \to 0 \ \ as \ \ k\to\infty \\ \to 0 \ \ as \ \ k\to\infty \\ \to 0 \ \ as \ \ k\to\infty \end{cases}$$

i.e., (4.1) is holds. Since, we have

$$\rho(x) = \frac{1}{(n^2-5n+6)}\chi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right)$$

then

$$\rho(x) = \frac{1}{(n^2-5n+6)}\chi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right)$$

$$= \begin{cases} \dfrac{\varepsilon}{4\left(n^2-5n+6\right)} \\ \dfrac{2\varepsilon\|x\|^s}{\left(n^2-5n+6\right)2^s} \\ \dfrac{2\varepsilon\|x\|^{ns}}{\left(n^2-5n+6\right)2^{ns}} \end{cases}$$

Also,

$$\frac{1}{\psi_i^2}\rho(\psi_i x)=\begin{cases}\dfrac{1}{\psi_i^2}\dfrac{\varepsilon}{(n^2-5n+6)}\\ \dfrac{1}{\psi_i^2}\dfrac{2\varepsilon\|x\|^s\psi_i^s}{(n^2-5n+6)2^s}\\ \dfrac{1}{\psi_i^2}\dfrac{2\varepsilon\|x\|^{ns}\psi_i^{ns}}{(n^2-5n+6)2^{ns}}\end{cases}=\begin{cases}\psi_i^{-2}\rho(x)\\ \psi_i^{s-2}\rho(x)\\ \psi_i^{ns-2}\rho(x)\end{cases}$$

for all $x\in X$. Hence the inequality (1.5) holds for following cases:

$L=2^{-2}$ if $i=0$ and $L=2^2$ if $i=1$

$L=2^{s-2}$ for $s<2$ if $i=0$ and $L=2^{2-s}$ for $s>2$ if $i=1$

$L=2^{ns-2}$ for $s<\dfrac{2}{n}$ if $i=0$ and $L=2^{2-ns}$ for $s>\dfrac{2}{n}$ if $i=1$

Now from (4.4), we prove the following cases.

Case 1. $L=2^{-2}$ if $i=0$

$$\|f(x)-G(x)\|\le\frac{L^{1-i}}{1-L}\rho(x)=\frac{2^{-2}}{1-2^{-2}}\frac{\varepsilon}{(n^2-5n+6)}=\frac{\varepsilon}{3(n^2-5n+6)}$$

Case 2. $L=2$ if $i=1$

$$\|f(x)-G(x)\|\le\frac{L^{1-i}}{1-L}\rho(x)=\frac{1}{1-2^2}\frac{\varepsilon}{(n^2-5n+6)}=\frac{\varepsilon}{-3(n^2-5n+6)}$$

Case 3. $L=2^{s-2}$ for $s<2$ if $i=0$

$$\|f(x)-G(x)\|\le\frac{L^{1-i}}{1-L}\rho(x)=\frac{2^{s-2}}{1-2^{s-2}}\frac{2\varepsilon\|x\|^s}{(n^2-5n+6)2^s}=\frac{2\varepsilon\|x\|^s}{(n^2-5n+6)(2^2-2^s)}$$

Case 4. $L=2^{2-s}$ for $s>2$ if $i=1$

$$\|f(x)-G(x)\|\le\frac{L^{1-i}}{1-L}\rho(x)=\frac{1}{1-2^{2-s}}\frac{2\varepsilon\|x\|^s}{(n^2-5n+6)2^s}=\frac{2\varepsilon\|x\|^s}{(n^2-5n+6)(2^s-2^2)}$$

Case 5. $L=2^{ns-2}$ for $s<\dfrac{2}{n}$ if $i=0$

$$\|f(x)-G(x)\|\le\frac{L^{1-i}}{1-L}\rho(x)=\frac{2^{ns-2}}{1-2^{ns-2}}\frac{2\varepsilon\|x\|^{ns}}{(n^2-5n+6)2^{ns}}=\frac{2\varepsilon\|x\|^{ns}}{(n^2-5n+6)(2^2-2^{ns})}$$

Case 6. $L=2^{2-ns}$ for $s>\dfrac{2}{n}$ if $i=1$

$$\|f(x)-G(x)\|\le\frac{L^{1-i}}{1-L}\rho(x)=\frac{1}{1-2^{2-ns}}\frac{2\varepsilon\|x\|^{ns}}{(n^2-5n+6)2^{ns}}=\frac{2\varepsilon\|x\|^{ns}}{(n^2-5n+6)(2^{ns}-2^2)}$$

Hence the proof is complete.

V. Fuzzy Stability Results

In this segment, the authors present basic definition in fuzzy normed space and investigate the fuzzy stability of the n-dimensional quadratic functional equation (1.5).

Definition 5.1 Let x be a real linear space. A function $F: X \times R \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $p, q \in R$

(N1) $F(x,c) = 0$ for $c \le 0$;

(N2) $x = 0$ if and only if $F(x,c) = 1$ for all $c > 0$;

(N3) $F(cx,q) = F\left(x, \frac{q}{|c|}\right) \quad if \quad c \ne 0$;

(N4) $F(x+y, p+q) \ge \min\{F(x,p), F(y,q)\}$;

(N5) $F(x,.)$ is a non-decreasing function on R and $\lim_{q\to\infty} F(x,q) = 1$;

(N6) for $x \ne 0,\ F(x,.)$ is continuous on R;

The pair (X,F) is called fuzzy normed linear space one may regard $F(x,q)$ as the truth value of the statement the norm of x is less than or equal to the real number q.

Definition 5.2 Let (X,F) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty}(x_n - x, q) = 1$ for all $t > 0$. In that case x is called the limit of the sequence x_n and we denote it by $F - \lim_{n\to\infty} x_n = x$.

Definition 5.3 A sequence $\{x_n\}$ be in x is called Cauchy if for each $\varepsilon > 0$ and each $q > 0$ there exists n_0 such that for all $n \ge n_0$ and all $r > 0$, we have $F(x_{n+r} - x_n, q) > 1 - \varepsilon$

Definition 5.4 Every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and fuzzy normed space is called a fuzzy Banach space.

In segment 6 and 7, assume that X, $(Z, F^{'})$ and $(Y, F^{'})$ are linear space, fuzzy normed space and fuzzy Banach space respectively. We define a function $Q: X \to Y$ by

$$Q(x_1, x_2, ..., x_n) = \sum_{\substack{i=1 \\ i\ne j\ne k}}^{n} f\left(-x_i - x_j - x_k + \sum_{\substack{i=1 \\ i\ne j\ne k}}^{n} x_i\right) - \left(\frac{n^3 - 15n^2 + 62n - 72}{6}\right)\sum_{\substack{i=1 \\ i\ne j}}^{n} f(x_i + x_j)$$
$$-\left(\frac{-n^4 + 17n^3 - 80n^2 + 136n - 72}{6}\right)\sum_{i=1}^{n} f(x_i)$$

for all $x_1, x_2, ..., x_n \in X$.

VI. Stability of the functional equation (1.5)- Direct Method

In this segment, we establish the stability of (1.5) in fuzzy Banach space using Direct Method.

Theorem 6.1 Let $\beta \in \{-1,1\}$. Let $\chi : X^n \to Z$ be a mapping with $0 < \left(\frac{d}{2^2}\right) < 1$

$$F^{'}\left(\chi\left(2^{\beta k} x_1, 2^{\beta k} x_2, ..., 2^{\beta k} x_n\right), r\right) \ge F^{'}\left(d^{\beta}(x, x, 0, ..., 0), r\right) \tag{6.1}$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F^{'}\left(\chi\left(2^{\beta k} x_1, 2^{\beta k} x_2, ..., 2^{\beta k} x_n\right), r\right) = 1 \tag{6.2}$$

for all $x_1, x_2, ..., x_n \in X$ and all $r > 0$. Suppose that a function $f: X \to Y$ satisfies the inequality

$$F\left(Q(x_1, x_2, ..., x_n), r\right) \ge F^{'}\left(\chi(x_1, x_2, ..., x_n), r\right) \tag{6.3}$$

for all $r > 0$ and $x_1, x_2, ..., x_n \in X$ the limit

$$G(x) = F - \lim_{k\to\infty} \frac{f\left(2^{\beta k} x\right)}{2^{2\beta k}} \quad (6.4)$$

exists for all $x \in X$ and the mapping $G : X \to Y$ is a unique quadratic mapping such that

$$F\left(f(x) - G(x), r\right) \geq F'\left(\chi(x,x,0,...,0), \left(n^2-5n+6\right) r \left|2^2-d\right|\right) \quad (6.5)$$

for all $x \in X$ and for all $r > 0$.

Proof. First assume that $\beta = 1$. Replacing $(x_1, x_2, ..., x_n)$ by $(x,x,0,...,0)$, in (6.3), we have

$$F\left(\left(\left(n^2-5n+6\right) f(2x) - 4\left(n^2-5n+6\right) f(x)\right), r\right) \geq F'\left(\chi(x,x,0,...,0), r\right) \quad (6.6)$$

for all $x \in X$ and for all $r > 0$. Replacing x by $2^k x$ in (6.6), we obtain

$$F\left(\frac{f\left(2^{k+1}x\right)}{2^2} - f\left(2^k x\right), \frac{r}{4\left(n^2-5n+6\right)}\right) \geq F'\left(\chi\left(2^k x, 2^k x, 0,...,0\right), r\right) \quad (6.7)$$

for all $x \in X$ and for all $r > 0$. Using (6.1), (N3) in (6.7), we have

$$F\left(\frac{f\left(2^{k+1}x\right)}{2^2} - f\left(2^k x\right), \frac{r}{4\left(n^2-5n+6\right)}\right) \geq F'\left(\chi\left(2^k x, 2^k x, 0,...,0\right), \frac{r}{d^k}\right) \quad (6.8)$$

for all $x \in X$ and for all $r > 0$, it is easy to verify from (6.8), that

$$F\left(\frac{f\left(2^{k+1}x\right)}{2^{2(k+1)}} - \frac{f\left(2^k x\right)}{2^k}, \frac{r}{4\left(n^2-5n+6\right)2^k}\right) \geq F'\left(\chi\left(2^k x, 2^k x, 0,...,0\right), \frac{r}{d^k}\right) \quad (6.9)$$

holds for all $x \in X$ and for all $r > 0$. Replacing r by $d^k r$ in (6.9)

$$F\left(\frac{f\left(2^{k+1}x\right)}{2^{2(k+1)}} - \frac{f\left(2^k x\right)}{2^k}, \frac{d^k r}{4\left(n^2-5n+6\right)2^k}\right) \geq F'\left(\chi\left(2^k x, 2^k x, 0,...,0\right), r\right) \quad (6.10)$$

for all $x \in X$ and for all $r > 0$, it is easy to see that

$$\frac{f\left(2^{k+1}x\right)}{2^{2(k+1)}} - f(x) = \sum_{i=0}^{k-1}\left[\frac{f\left(2^{i+1}x\right)}{2^{2(i+1)}} - \frac{f\left(2^i x\right)}{2^{2i}}\right] \quad (6.11)$$

for all $x \in X$. From the equations (6.10) and (6.11), we get

$$\begin{aligned} F\left(\frac{f\left(2^k x\right)}{2^{2k}} - f(x), \sum_{i=0}^{k-1}\frac{d^i r}{4\left(n^2-5n+6\right)2^{2i}}\right) &\geq \min \bigcup_{i=1}^{k-1}\left\{\frac{f\left(2^{i+1}x\right)}{2^{2(i+1)}} - \frac{f\left(2^i x\right)}{2^{2i}}, \frac{d^i r}{4\left(n^2-5n+6\right)2^{2i}}\right\} \\ &\geq \min \bigcup_{i=1}^{k-1} F'\left(\chi(x,x,0,...,0), r\right) \\ &\geq F'\left(\chi(x,x,0,...,0), r\right) \quad (6.12) \end{aligned}$$

for all $x \in X$ and for all $r > 0$. Replacing x by $2^m x$ in (6.12) and using (6.1) and (N3), we obtain

$$F\left(\frac{f\left(2^{k+m}x\right)}{2^{2(k+m)}} - \frac{f\left(2^m x\right)}{2^{2m}}, \sum_{i=0}^{m+k-1}\frac{d^i r}{4\left(n^2-5n+6\right)2^{2i}}\right) \geq F'\left(\chi(x,x,0,...,0), \frac{r}{d^m}\right) \quad (6.13)$$

for all $x \in X$ and for all $r > 0$. And all $m, k \geq 0$. Replacing r by $d^m r$ in (6.13), we get

$$F\left(\frac{f\left(2^{k+m}x\right)}{2^{2(k+m)}}-\frac{f\left(2^{m}x\right)}{2^{2m}},\sum_{i=m}^{m+k-1}\frac{d^{i}r}{4\left(n^{2}-5n+6\right)2^{2i}}\right)\geq F^{'}\left(\chi\left(x,x,0,...,0\right),r\right) \tag{6.14}$$

for all $x\in X$ and for all $r>0$. And all $m,k\geq 0$. Using (N3) in (6.13), we have

$$F\left(\frac{f\left(2^{k+m}x\right)}{2^{2(k+m)}}-\frac{f\left(2^{m}x\right)}{2^{2m}},r\right)\geq F^{'}\left(\chi\left(x,x,0,...,0\right),\frac{r}{\sum_{i=m}^{m+k-1}\frac{d^{i}}{4\left(n^{2}-5n+6\right)2^{2i}}}\right) \tag{6.15}$$

for all $x\in X$ and for all $r>0$. And all $m,k\geq 0$. Since $0<d<2^{2}$ and $\sum_{i=0}^{k}\left(\frac{d}{2^{2}}\right)^{i}<\infty$. The Cauchy criterion for convergence and (N5) implies that $\left\{\frac{f\left(2^{k}x\right)}{2^{2k}}\right\}$ is a Cauchy sequence in $\left(Y,F^{'}\right)$ is a fuzzy Banach space. This sequence converges to some point $G\left(x\right)\in Y$ so one can define the mapping $G:X\rightarrow Y$ by

$$G\left(x\right)=N-\lim_{k\to\infty}\frac{f\left(2^{k}x\right)}{2^{2k}}$$

for all $x\in X$. Letting $m=0$ in (6.15), we receive

$$F\left(\frac{f\left(2^{k}x\right)}{2^{2k}}-f\left(x\right),r\right)\geq F^{'}\left(\chi\left(x,x,0,...,0\right),\frac{r}{\sum_{i=0}^{k-1}\frac{d^{i}}{4\left(n^{2}-5n+6\right)2^{2i}}}\right) \tag{6.16}$$

for all $x\in X$. Letting $k\to\infty$ in (6.16) and using (N6), we have

$$F\left(f\left(x\right)-G\left(x\right),r\right)\geq F^{'}\left(\chi\left(x,x,0,...,0\right),\left(n^{2}-5n+6\right)r\left(2^{2}-d\right)\right)$$

for all $x\in X$ and for all $r>0$. To prove G satisfies (1.5), replacing $\left(x_{1},x_{2},...,x_{n}\right)$ by $\left(2^{k}x_{1},2^{k}x_{2},...,2^{k}x_{n}\right)$ in (6.3), we get

$$F\left(\frac{1}{2^{2k}}Q\left(2^{k}x_{1},2^{k}x_{2},...,2^{k}x_{n}\right),r\right)\geq F^{'}\left(\chi\left(2^{k}x_{1},2^{k}x_{2},...,2^{k}x_{n}\right),2^{2k}r\right) \tag{6.17}$$

for all $r>0$ and all $x_{1},x_{2},...,x_{n}\in X$. Now

$$F\left(\left(\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}G\left(-x_{i}-x_{j}-x_{k}+\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}x_{i}\right)\right)-\left(\frac{n^{3}-15n^{2}+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n}G\left(x_{i}+x_{j}\right)-\left(\frac{-n^{4}+17n^{3}-80n^{2}+136n-72}{6}\right)\sum_{i=1}^{n}G\left(x_{i}\right),r\right)$$

$$\geq\min\left\{F\left(\left(\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}G\left(-x_{i}-x_{j}-x_{k}+\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}x_{i}\right)-\frac{1}{2^{2k}}\left(\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}f\left(2^{k}\left(-x_{i}-x_{j}-x_{k}+\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}x_{i}\right)\right)\right)\right),\frac{r}{4}\right),\right.$$

$$F\left(\left(\frac{n^3-15n^2+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n}G\left(x_i+x_j\right)-\frac{1}{2^{2k}}\left(\frac{n^3-15n^2+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n}f\left(2^k\left(x_i+x_j\right)\right),\frac{r}{4}\right),$$

$$F\left(\left(\frac{-n^4+17n^3-80n^2+136n-72}{6}\right)\sum_{i=1}^{n}G\left(x_i\right)-\frac{1}{2^{2k}}\left(\frac{-n^4+17n^3-80n^2+136n-72}{6}\right)\sum_{i=1}^{n}f\left(2^k x_i\right),\frac{r}{4}\right),$$

$$F\left(\frac{1}{2^{2k}}\left(\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}f\left(2^k\left(-x_i-x_j-x_k+\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}x_i\right)\right)\right)-\frac{1}{2^{2k}}\left(\frac{n^3-15n^2+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n}f\left(x_i+x_j\right)\right.$$

$$\left.-\frac{1}{2^{2k}}\left(\frac{-n^4+17n^3-80n^2+136n-72}{6}\right)\sum_{i=1}^{n}f\left(2^k x_i\right),\frac{r}{4}\right)\Bigg\} \qquad (6.18)$$

for all $x_1,x_2,...,x_n\in X$ and all $r>0$, using (6.7) and (N5) in (6.18), we see that

$$F\left(\left(\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}G\left(-x_i-x_j-x_k+\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}x_i\right)\right)-\left(\frac{n^3-15n^2+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n}G\left(x_i+x_j\right)-\left(\frac{-n^4+17n^3-80n^2+136n-72}{6}\right)\sum_{i=1}^{n}G\left(x_i\right),r\right)$$

$$\geq \min\left\{1,1,1,F'\left(\chi\left(2^k x_1,2^k x_2,...,2^k x_n\right),2^{2k}r\right)\right\}$$

$$\geq F'\left(\chi\left(2^k x_1,2^k x_2,...,2^k x_n\right),2^{2k}r\right) \qquad (6.19)$$

for all $x_1,x_2,...,x_n\in X$ and all $r>0$. Letting $k\to\infty$ in (6.19) and using (6.2), we have

$$F\left(\left(\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}G\left(-x_i-x_j-x_k+\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}x_i\right)\right)-\left(\frac{n^3-15n^2+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n}G\left(x_i+x_j\right)-\left(\frac{-n^4+17n^3-80n^2+136n-72}{6}\right)\sum_{i=1}^{n}G\left(x_i\right),r\right)$$

$$=1$$

for all $x_1,x_2,...,x_n\in X$ and all $r>0$. Using (N2) in the above inequality gives

$$\left(\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}G\left(-x_i-x_j-x_k+\sum_{\substack{i=1\\ i\neq j\neq k}}^{n}x_i\right)\right)=\left(\frac{n^3-15n^2+62n-72}{6}\right)\sum_{\substack{i=1\\ i\neq j}}^{n}G\left(x_i+x_j\right)+\left(\frac{-n^4+17n^3-80n^2+136n-72}{6}\right)\sum_{i=1}^{n}G\left(x_i\right)$$

for all $x_1,x_2,...,x_n\in X$. Hence G satisfies the quadratic functional equation (1.5). In order to prove $G(x)$ is unique. We let $G'(x)$ be another quadratic functional equation satisfying (1.5) and (6.5). Hence

$$F\left(G(x)-G'(x),r\right)=F\left(\frac{G\left(2^k x\right)}{2^{2k}}-\frac{G'\left(2^k x\right)}{2^{2k}}\right)$$

$$\geq \min\left\{F\left(\frac{G\left(2^k x\right)}{2^{2k}}-\frac{f\left(2^k x\right)}{2^{2k}},\frac{r}{2}\right),F\left(\frac{f\left(2^k x\right)}{2^{2k}}-\frac{G'\left(2^k x\right)}{2^{2k}},\frac{r}{2}\right)\right\}$$

$$\geq F'\left(\chi\left(2^k x,2^k x,0,...,0\right),\frac{\left(n^2-5n+6\right)2^{2k}r\left(2^2-d\right)}{2}\right)$$

$$\geq F'\left(\chi\left(2^k x,2^k x,0,...,0\right),\frac{\left(n^2-5n+6\right)2^{2k}r\left(2^2-d\right)}{2d^k}\right)$$

for all $x \in X$ and for $r > 0$. Since,

$$\lim_{k\to\infty} \frac{\left(n^2-5n+6\right)2^{2k} r\left(2^2-d\right)}{2d^k} = 0$$

we obtain

$$F'\left(\chi\left(2^k x, 2^k x, 0, ..., 0\right), \frac{\left(n^2-5n+6\right)2^{2k} r\left(2^2-d\right)}{2d^k}\right) = 1$$

Thus $F\left(G(x)-G'(x), r\right) = 1$ for all $x \in X$ and for $r > 0$. Hence $G(x) = G'(x)$. Therefore $G(x)$ is unique.

For $\beta = -1$, we can prove the result by a similar method. This completes the proof of the Theorem.

Corollary 6.2 Suppose that the function $f : X \to Y$ satisfies the inequality

$$F\left(Q\left(x_1, x_2, ..., x_n\right), r\right) \ge \begin{cases} F'\left(\varepsilon, r\right)' \\ F'\left(\varepsilon \sum_{i=1}^{n} \|x_i\|^s, r\right), \\ F'\left(\varepsilon\left(\sum_{i=1}^{n} \|x_i\|^{ns} + \prod_{i=1}^{n} \|x_i\|^s\right), r\right), \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$ and all $r > 0$, where ε, s are constants. Then there exists a unique quadratic mapping $G : X \to Y$ such that

$$F\left(f(x)-G(x), r\right) \ge \begin{cases} F'\left(\varepsilon, 3r\left(n^2-5n+6\right)\right) \\ F'\left(2\varepsilon\|x\|^s, r\left(n^2-5n+6\right)\left(2^2-2^s\right)\right) \\ F'\left(2\varepsilon\|x\|^{ns}, r\left(n^2-5n+6\right)\left(2^2-2^{ns}\right)\right) \end{cases}$$

for all $x \in X$ and for $r > 0$.

VII. Stability of the Functional Equation (1.5) – Fixed Point Method

In this segment, the authors investigate the generalized Ulam-Hyers stability of the functional equation (1.5) in fuzzy normed space using fixed point method.

For to prove the stability result we define the following μ_i is a constant such that

$$\mu_i = \begin{cases} 2 & if \quad i = 0 \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that $\Omega = \left\{p \setminus p : x \to y, p(0) = 0\right\}$.

Theorem 7.1 Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^n \to Z$ with condition

$$\lim_{k\to\infty} F'\left(\chi\left(\psi^k x_1, \psi^k x_2, ..., \psi^k x_n\right), \psi^k r\right) = 1 \tag{8.1}$$

for all $x_1, x_2, ..., x_n \in X$, $r > 0$ and satisfying the inequality

$$F\left(Q(x_1, x_2, ..., x_n), r\right) \ge F'\left(\chi\left(x_1, x_2, ..., x_n\right), r\right) \tag{8.2}$$

for all $x_1, x_2, ..., x_n \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \rho(x)$ has the property

$$F'\left(L\frac{1}{\psi_i^2}\rho(\psi_i x), r\right) = F'\left(\rho(x), r\right) \quad (8.3)$$

for all $x \in X$ and $r > 0$. Then there exists unique quadratic function $G: X \to Y$ satisfying the functional equation (1.5) and

$$F\left(f(x) - G(x), r\right) \ge F'\left(\frac{L^{1-i}}{1-L}\rho(x), r\right)$$

for all $x \in X$ and $r > 0$.

Proof. Let d be a general metric on Ω, such that

$$d(p,q) = \inf\left\{k \in (0,\infty) / F\left(p(x) - q(x), r\right) \ge F'\left(\rho(x), kr\right),\ x \in X, r > 0\right\}$$

It is easy to see that (Ω, χ) is complete. Define $T: \Omega \to \Omega$ by $T_p(x) = \frac{1}{\psi_i^2} p(\psi_i x),\ \ \forall\ x \in X.$

For $p, q \in \Omega$, we get

$$d(p,q) = k \Rightarrow F\left(p(x) - q(x)\right) \ge F'\left(\rho(x), kr\right)$$
$$\Rightarrow F\left(\frac{p(\psi_i x)}{\psi_i^2} - \frac{q(\psi_i x)}{\psi_i^2}, r\right) \ge F'\left(\rho(\psi_i x), k\psi_i r\right) \quad (7.4)$$
$$\Rightarrow F\left(T_p(x) - T_q(x), r\right) \ge F'\left(\rho(\psi_i x), k\psi_i r\right)$$
$$\Rightarrow F\left(T_p(x) - T_q(x), r\right) \ge F'\left(\rho(x), kLr\right)$$
$$\Rightarrow d\left(T_p(x) - T_q(x), r\right) \ge kL$$
$$\Rightarrow d\left(T_p - T_q, r\right) \ge kd(0,1)\ \ \forall p, q \in \Omega.$$

Therefore, T is strictly contractive mapping on Ω with Lipschitz constant L, replacing $(x_1, x_2, ..., x_n)$ by $(x, x, 0, ..., 0)$ in (7.2), we get

$$F\left(\left(n^2 - 5n + 6\right) f(2x) - 4\left(n^2 - 5n + 6\right) f(x), r\right) \ge F'\left(\chi(x, x, 0, ..., 0), r\right) \quad (7.5)$$

for all $x \in X$ and $r > 0$. Using (N3) in (7.5), we have

$$F\left(\frac{f(2x)}{2^2} - f(x), r\right) \ge F'\left(\frac{1}{4\left(n^2 - 5n + 6\right)}\chi(x, x, 0, ..., 0), r\right) \quad (7.6)$$

for all $x \in X$ and $r > 0$ with the help of (7.3), when $i = 0$. It follows from (7.6) that

$$\Rightarrow F\left(\frac{f(2x)}{2^2} - f(x), r\right) \ge F'\left(L\rho(x), r\right)$$
$$\Rightarrow d\left(T_f(x), r\right) \ge L = L^1 = L^{1-i} \quad (7.7)$$

Replacing x by $\frac{x}{2}$ in (7.5), we receive

$$F\left(f(x) - 4f\left(\frac{x}{2}\right), r\right) \ge F'\left(\frac{1}{\left(n^2 - 5n + 6\right)}\chi\left(\frac{x}{2}, \frac{x}{2}, 0, ..., 0\right), r\right) \quad (7.8)$$

for all $x \in X$ and $r > 0$ when $i = 1$ it follows from (7.8), we arrive

$$\Rightarrow F\left(f(x) - 4f\left(\frac{x}{2}\right), r\right) \ge F'\left(\rho(x), r\right)$$

$$\Rightarrow T\left(f-T_f\right)\le 1 = L^0 = L^{1-i} \tag{7.9}$$

Then from (7.7) and (7.9), we get

$$\Rightarrow T\left(f,T_f\right)\le L^{1-i} < \infty .$$

Now from the fixed point alternative in both cases it follows that there exists a fixed point G of T in Ω such that

$$G(x) = N-\lim_{k\to\infty}\frac{f\left(\psi^k x\right)}{\psi^{2k}} \tag{7.10}$$

for all $x\in X$ and $r>0$. Replacing $\left(x_1,x_2,...,x_n\right)$ by $\left(\psi_i^k x_1,\psi_i^k x_2,...,\psi_i^k x_n\right)$ in (7.2), we get

$$F\left(\frac{1}{\psi_i^{2k}}Q\left(\psi_i^k x_1,\psi_i^k x_2,...,\psi_i^k x_n\right),r\right)\ge F'\left(\chi\left(\psi_i^k x_1,\psi_i^k x_2,...,\psi_i^k x_n\right),\psi_i^{2k}r\right)$$

for all $r>0$ and all $x_1,x_2,...,x_n\in X$. By proceeding some procedure in the theorem (6.7), we can prove the function $G:X\to Y$ is quadratic and its satisfies the functional equation (1.5) by a fixed point alternative. Since G is unique fixed point of T in the set $\Delta=\left\{f\in\Omega / d\left(f,G\right)<\infty\right\}$. Therefore, G is a unique function such that

$$F\left(f(x)-G(x),r\right)\ge F'\left(\rho(x),kr\right) \tag{7.11}$$

for all $x\in X$ and $r>0$. Again, using the fixed point alternative, we get

$$d\left(f,G\right)\le\frac{1}{1-L}d\left(f,Tf\right)$$

$$\Rightarrow d\left(f,G\right)\le\frac{L^{1-i}}{1-L}$$

$$\Rightarrow F\left(f(x)-G(x),r\right)\ge F'\left(\rho(x)\frac{L^{1-i}}{1-L},r\right) \tag{7.12}$$

This completes the proof of the Theorem.
The following corollary is an immediate consequence of Theorem 7.1 concerning the stability of (1.5).

Corollary 7.2 Suppose that a function $f:X\to Y$ satisfies the inequality

$$F\left(Q\left(x_1,x_2,...,x_n\right),r\right)\ge\begin{cases}F'\left(\varepsilon,r\right),\\ F'\left(\varepsilon\left\{\sum_{i=1}^{n}\|x_i\|^s\right\},r\right),\\ F'\left(\varepsilon\left\{\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right\},r\right),\end{cases} \tag{7.13}$$

for all $x_1,x_2,...,x_n\in X$ and $r>0$, where ε, s are constants with $\varepsilon>0$. Then there exists an unique quadratic function $G:X\to Y$ such that

$$F\left(f(x)-G(x),r\right)\le\begin{cases}F'\left(\varepsilon,|3|\left(n^2-5n+6\right)r\right)\\ F'\left(2\varepsilon\|x\|^s,\left(n^2-5n+6\right)\left|2^2-2^s\right|r\right) & ;\ s\ne 2\\ F'\left(2\varepsilon\|x\|^{ns},\left(n^2-5n+6\right)\left|2^2-2^{ns}\right|r\right) & ;\ s\ne\dfrac{2}{n}\end{cases} \tag{7.14}$$

for all $x\in X$ and $r>0$.

Proof. Setting

$$\chi(x_1,x_2,...,x_n) \le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{n}\|x_i\|^s\right\}, \\ \varepsilon\left\{\prod_{i=1}^{n}\|x_i\|^s + \sum_{i=1}^{n}\|x_i\|^{ns}\right\}, \end{cases}$$

for all $x_1, x_2,..., x_n \in X$. Then

$$F'\left(\chi\left(\psi_i^k x_1, \psi_i^k x_2,...,\psi_i^k x_n\right), \psi_i^{2k} r\right) = \begin{cases} F'\left(\varepsilon, \psi_i^{2k} r\right), \\ F'\left(\varepsilon\left\{\sum_{i=1}^{n}\|x_i\|^s\right\}, \psi_i^{(2-s)k} r\right), \\ F'\left(\varepsilon\left\{\prod_{i=1}^{n}\|x_i\|^s + \sum_{i=1}^{n}\|x_i\|^{ns}\right\}, \psi_i^{(2-ns)k} r\right) \end{cases}$$

$$= \begin{cases} \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty \end{cases}$$

i.e., (7.1) is holds. Since, we have

$$\rho(x) = \frac{1}{(n^2 - 5n + 6)} \chi\left(\frac{x}{2}, \frac{x}{2}, 0,...,0\right)$$

has the property

$$F'\left(L\frac{1}{\psi_i^2}\rho(\psi_i x), r\right) = F'\left(\rho(x), r\right)$$

for all $x \in X$ and $r > 0$. Hence

$$F'\left(\rho(x), r\right) = F'\left(\chi\left(\frac{x}{2}, \frac{x}{2}, 0,...,0\right), \left(n^2 - 5n + 6\right) r\right)$$

$$= \begin{cases} F'\left(\varepsilon, 4\left(n^2 - 5n + 6\right) r\right) \\ F'\left(2\varepsilon\|x\|^s, \left(n^2 - 5n + 6\right) 2^s r\right) \\ F'\left(2\varepsilon\|x\|^{ns}, \left(n^2 - 5n + 6\right) 2^{ns} r\right) \end{cases}$$

Now

$$F'\left(\frac{1}{\psi_i^2}\rho(\psi_i x), r\right) = \begin{cases} F'\left(\frac{\varepsilon}{\psi_i^2}, \left(n^2-5n+6\right)r\right) \\ F'\left(\frac{2\varepsilon\|x\|^s \psi_i^s}{\psi_i^2 2^s}, \left(n^2-5n+6\right)r\right) \\ F'\left(\frac{2\varepsilon\|x\|^{ns} \psi_i^{ns}}{\psi_i^2 2^{ns}}, \left(n^2-5n+6\right)r\right) \end{cases} = \begin{cases} \psi_i^{-2}\rho(x) \\ \psi_i^{s-2}\rho(x) \\ \psi_i^{ns-2}\rho(x) \end{cases}$$

for all $x \in X$. Now from the following cases for the conditions

$L = 2^{-2}$ if $i = 0$ and $L = 2^2$ if $i = 1$

$L = 2^{s-2}$ for $s > 2$ if $i = 0$ and $L = 2^{2-s}$ for $s < 2$ if $i = 1$

$L = 2^{ns-2}$ for $s > \frac{2}{n}$ if $i = 0$ and $L = 2^{2-ns}$ for $s < \frac{2}{n}$ if $i = 1$

Case 1. $L = 2^{-2}$ if $i = 0$

$$F\left(f(x) - G(x), r\right) \le F' = \left(\frac{L^{1-i}}{1-L}\rho(x), r\right) = \left(\frac{2^{-2}}{1-2^{-2}}\frac{\varepsilon}{\left(n^2-5n+6\right)}, r\right) = \left(\varepsilon, 3\left(n^2-5n+6\right)r\right)$$

Case 2. $L = 2$ if $i = 1$

$$F\left(f(x) - G(x), r\right) \le F' = \left(\frac{L^{1-i}}{1-L}\rho(x), r\right) = \left(\frac{1}{1-2^{2}}\frac{\varepsilon}{\left(n^2-5n+6\right)}, r\right) = \left(\varepsilon, -3\left(n^2-5n+6\right)r\right)$$

Case 3. $L = 2^{s-2}$ for $s > 2$ if $i = 0$

$$F\left(f(x) - G(x), r\right) \le F' = \left(\frac{L^{1-i}}{1-L}\rho(x), r\right) = \left(\frac{2^{s-2}}{1-2^{s-2}}\frac{2.\varepsilon\|x\|^s}{\left(n^2-5n+6\right)2^s}, r\right) = \left(2\varepsilon\|x\|^s, \left(n^2-5n+6\right)\left(2^2-2^s\right)r\right)$$

Case 4. $L = 2^{2-s}$ for $s < 2$ if $i = 1$

$$F\left(f(x) - G(x), r\right) \le F' = \left(\frac{L^{1-i}}{1-L}\rho(x), r\right) = \left(\frac{1}{1-2^{2-s}}\frac{2.\varepsilon\|x\|^s}{\left(n^2-5n+6\right)2^s}, r\right) = \left(2\varepsilon\|x\|^s, \left(n^2-5n+6\right)\left(2^s-2^2\right)r\right)$$

Case 5. $L = 2^{ns-2}$ for $s > \frac{2}{n}$ if $i = 0$

$$F\left(f(x) - G(x), r\right) \le F' = \left(\frac{L^{1-i}}{1-L}\rho(x), r\right) = \left(\frac{2^{ns-2}}{1-2^{ns-2}}\frac{2.\varepsilon\|x\|^{ns}}{\left(n^2-5n+6\right)2^{ns}}, r\right) = \left(2\varepsilon\|x\|^{ns}, \left(n^2-5n+6\right)\left(2^2-2^{ns}\right)r\right)$$

Case 6. $L = 2^{2-ns}$ for $s < \frac{2}{n}$ if $i = 1$

$$F\left(f(x) - G(x), r\right) \le F' = \left(\frac{L^{1-i}}{1-L}\rho(x), r\right) = \left(\frac{1}{1-2^{2-ns}}\frac{2.\varepsilon\|x\|^{ns}}{\left(n^2-5n+6\right)2^{ns}}, r\right) = \left(2\varepsilon\|x\|^{ns}, \left(n^2-5n+6\right)\left(2^{ns}-2^2\right)r\right)$$

Hence the proof is complete.

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