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What Does A Tensor Serve?

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Abstract: *Explaining what a tensor is is not an easy task; worse still is trying to visualize it. However, in a very simple way, we can say that a tensor can be defined as a set of entities that satisfy some basic rules, something similar to the vectors. The vectors satisfy the rules of the vector space, while the tensors obey the rules of a tensorial space. The vector space is contained in the tensorial space. The mathematical entity called a tensor is a generalization of the concept of a vector. Aliases, numbers, vectors and matrices, are examples of tensors, and the difference between them is that each has a certain order. Number is a "order tensor 0", vector is a "order tensor 1", arrays are "order 2 tensors" and so on. The whole modern formulation of physics is based on tensor calculus, for these mathematical entities better describe the physical quantities in question, just as a vector better describes a displacement than a scalar number would describe. In fact a tensor serves, mathematically, to simplify a physical information.*

Keywords: *tensor, physics, applications*

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I. Introduction

The word tensor[1] was introduced in 1846 by William Rowan Hamilton. It was used in its present meaning by Woldemar Voigt in 1899. The tensorial calculation was developed around 1890 by Gregory Ricci-Curbastro under the title of Absolute Differential Calculus. In the 20th century the subject became known as tensor analysis and gained wider acceptance with the introduction of Einstein's theory of general relativity over 1915. Tensors are also used in other fields as the mechanics of continuity. To better clarify the mathematical definition of tensor, we can use the definition of NEARING[2], for whom the tensors have to do with functional. In general, a functional is a function that associates another function with a scalar. On many occasions a functional is associated with the mapping of a set of functions in a set of scalars. Still using the concepts of NEARING, a functional is a function that maps vectors in scalars. Even in many applications, these functions can be seen as vectors. Therefore, this second definition is more general than the first. We can also make an analogy between the functions of a vector and a tensor. Tensors can also be defined as magnitudes used to generalize the notion of scalars, vectors and matrices. Like such entities, a tensor is a form of representation associated with a set of operations such as sum and product. The order of a tensor is the dimensionality of the matrix used to represent it. The tensor can be represented by an array of nine elements, sixteen elements, and so on. A tensor of order n in a space with three dimensions has 3n components.

SECOND ORDER TENSOR

In analogy to a vector that requires three components to be specified, a tensor of order 2 needs nine components

$$\tau_{ij} = \begin{pmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix} \quad (1)$$

SOARES[4] writes that the tensors used in the General Theory of Relativity (TRG) are tensors of order 0,1 and 2, and space is 4-dimensional space-time, three spatial coordinates, and temporal coordinate. Thus, the second-order tensors of the TRG have, in principle, 16 components, and real physical problems impose constraints of symmetry that reduce the really necessary components to 10. The first order tensors are the vectors of the TRG, called quadrivetors and have 4 components.

An example of the second-order tensors is the tensor of the voltages. To illustrate this[3], let us consider a cubic element of a given material, which may be a fluid, for example. Each face of this hypothetical cube can be specified by the direction that is perpendicular to it and by the direction out of the cube. Suppose that on each face of this cube there is an acting force imposed by the remainder of the material, with components in the x, y, and z directions on each face of the cube. The quotient of this force by the area of the face on which it is applied is defined as tension. For an infinitely small cubic element we have the voltage at one point, which will be equal for each parallel face. This voltage in question has two directions, that of the face and that of the force component on that face. Thus τ_{xy} would mean the value of the stress on the face perpendicular to the x-axis, in the direction parallel to the y-axis. The complete tensor would still have the components τ_{xx}, τ_{xy}, τ_{xz}, τ_{yy}, τ_{yx}, $\tau_{yz,}$ τ_{zx}, τ_{zy} τ_{zz} . In a more compact way it is said that the tensor of the tensions is a matrix of dimension (3x3) where all the tensions are placed in an infinitesimal element in the volume element below, that is,

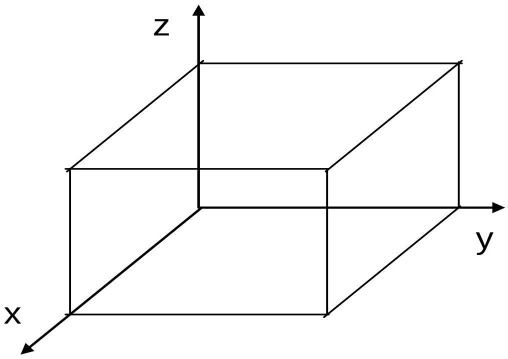

Figure 1 - Infinitesimal element in the volume element in the material

Each one of the six faces has a direction. A force acting on any face can act in the x, y and z directions. The force per unit face area acting in the x direction on that face is the stress τ_{yx} (first face, second stress). The forces per unit face area acting in the y and z directions on that face are the stresses τ_{yy} and τ_{yz}. Here τ_{yy} is a normal stress (acts normal, or perpendicular to the face) and τ_{yx} and τ_{yz} are shear stresses (act parallel to the face). Shear stresses similarly reverse sign on the opposite face face are the stresses τ_{yy} and τ_{yz}. Then, have the stress tensor τ, com 9 components.

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{pmatrix} \quad (2)$$

Following an analogous reasoning we can construct tensors of order greater than two. Second order tensors can be represented by square matrices, third order matrices by three-dimensional matrices, and so on. In relativity, in addition to the spatial components, the temporal component is also considered in the so-called quadritensors. An electric charge generates a field of electromagnetic tensors that results in interaction at each point of the electric field and the magnetic field. In general relativity theory[5], for example, we need to measure the curvature of spacetime, whose components are (t, x, y, z), so we need to use a magnitude that must have four indices time, length, width and height. This would be done by a tensor of order 4, i.e. a number with four indices.

SOARES[4], writes that space-time curvature is given, mathematically, by Einstein's tensor $G_{\mu\nu}$:

$$\mathbf{G}_{\mu\nu} = \mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} \quad (3)$$

With the indexes μ and ν taking the values of 0,1,2 and 3. The tensor $R_{\mu\nu}$ is called the Ricci tensor, formed from the Riemann curvature tensor, which is a tensor of order 4, being the most general to describe the curvature of any space of n any dimensions. The tensor $g_{\mu\nu}$ is the tensor of the metric of space-time and makes, in Einstein's equations, the role of the field. Still, according to SOARES, the matter and energy part of the

Einstein equations is given by the energy-moment tensor $T_{\mu\nu}$. The complete Einstein field equations then have the following compact form

$$G_{\mu\nu} = -kT_{\mu\nu} \tag{4}$$

where $k = 8\pi G/c^4$ is Einstein's gravitational constant, G is the universal gravitational constant, and c is the velocity of light in the vacuum. Note that in Newtonian language gravity is a force and in Einsteinian language it is itself the curvature of space-time, that is, there is no action at a distance so feared by Newton and many physicists. Now the very distribution of energy matter generates the curvature. In relation to matter and energy of Einstein's equations, they are described by the energy-momentum tensor (5). This is the tensor that describes the energetic activity in space. The moment-energy tensor quantitatively supplies the densities and flows of energy and momentum generated by the sources present in space and which will determine the geometry of space-time. The components of the energy-momentum tensor are as follows: material and energy density, energy fluxes, moment component densities, and current component fluxes, which are shear stresses and stresses. The energy-moment tensor $T_{\mu\nu}$, in relativity, is a symmetric tensor that describes the flow of the μ component of the quadri-moment pμ across the hyper-surface with constant xν. This tensor is of great use because it can be written for any physical object containing energy, whether it is described by a system of particles or fields. According to BEZERRA DE MELLO[6] the tensor energy moment is given by

$$\begin{aligned} \langle T_{\mu\nu}(x)\rangle \; &= \; \lim_{x'\to x} \partial_{\mu'}\partial_\nu G(x,x') + \left[(\xi - 1/4)\, g_{\mu\nu}\Box \right. \\ &- \left. \xi\nabla_\mu\nabla_\nu - \xi R_{\mu\nu}\right]\langle\phi^2(x)\rangle \; . \end{aligned} \tag{5}$$

Where ξ is the coupling constant of the field with the geometry, $R_{\mu\nu}$ is the tensorial field $g_{\mu\nu}$ is the fundamental metric tensor.

The energy-moment tensor describes the distribution and flow of energy and momentum due to the presence and movement of matter and radiation in a space-time region. In any case, in a region of interest, the components described energy density, the energy flux in all directions, divided by c (equivalently, the density of the corresponding components of the impulse, multiplied by c), and the fluxes of components of the moment in all directions.

In a simplified way, the moment energy tensor for electromagnetic fields is given by:

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{6}$$

Where $F^{\mu\upsilon}$ is the electromagnetic tensor and $\eta_{\mu\upsilon}$ is the Minkowski metric tensor.

In the relativistic dynamics[7] of the continuous means, for a uniform system the expected value of the energy-moment tensor assumes the following form

$$\langle T_{\mu\nu}\rangle = (\varepsilon + P)u_\mu u_\nu - Pg_{\mu\nu} \tag{7}$$

where in the previous formula we have:

is the quadrivector velocity that describes the movement.

is the fundamental metric tensor.

$g_{\mu\nu}$ e energy density and P is the pressure

oment energy tensor can be written by a 4x4 matrix given by

$$T_{\mu\nu} \; = \; \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} \tag{9}$$

In this nomenclature, we have:

T00 is the volumetric density of energy. T10, T20, T30 are the moment densities. T01, T02, T03 are energy flows. T21, T31, T32 are moment flows. T12, T13, T23 are terms of viscosity. T11, T22, T33 are terms referring to pressure.

The matrix formula of the electromagnetic tensor is given by:

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

where Ex Ey Ez are the components of electric field vector, Bx By Bz are components of magnetic field vector and c is the speed of light.

II. Final Considerations

Sometimes it is hard not to think in terms of tensors and their associated concepts, but geometrical considerations and physical applications help to make the subject a little more attractive. This article is an attempt to record the first notions about tensors and give some examples. To cite an illustrative case, we can say that the mathematical description of physical laws, to be valid, must be independent of the coordinate system employed: mathematical equations that express the laws of nature must be covariant, that is, invariant in their form under changes coordinates. It is exactly the fulfillment of this requirement that leads physicists to the study of Tensorial Calculus, of capital importance in the General Theory of Relativity and very useful in several other branches of Physics. Tensors are important in physics because they provide a concise mathematical structure for formulating and solving physics problems in areas such as elasticity, fluid mechanics, electromagnetism, general relativity, and more. Tensors are incredibly useful tools, particularly when describing things in higher dimensions. Other important considerations from the point of view of concepts and applications can be found in SOKOLNIKOFF(8).

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