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The Unique Fixed Point That Satisfies the φ-Shrink Type Mapping in b_2-Metric Space

Ke-Xia Yang[1], Hong-Dan Yin[1], Yi-Hao Sheng[1], Hui Xu[1]
[1]Department of Mathematics, College of Sciences, Yanbian University, Yanji 133002, China
Corresponding Author: Ke-Xia Yang

Abstract: *is proved that there is a unique common fixed point for the mapping in the b_2-metric space under the condition of satisfying such contraction type. We raised a weaker five-element function class and construct the sequence by shrinking conditions.The results of this paper generalize the theory of fixed points in many Fixed point theories.*

Key words: *b_2-metric space five-element function class shrink condition Cauchy sequence fixed point*

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I. Introduction

In the past half century, the theory of fixed point has made considerable progress.Many scholars apply it to the fields of nonlinear integral equations, functional differential equations, etc. in the Banach Space and achieved a series of achievements. In recent years, with the development of nonlinear functional analysis, the fixed point theory has been further developed.Many scholars have obtained some useful conclusions by promoting the Metric space.Among them, the most classic result is the concept of 2-metric space introduced by S.Gahler in 1963.There are also some scholars who have established nonlinear shrinkage by improving shrinkage conditions, and have generalized the fixed point theorems in many metric spaces.

It is worth noting that generalized metric spaces such as 2-metric space and b-metric in [1] space are not necessarily continuous for their arguments.To solve this problem, many scholars have conducted research.Until 2014, Zead Mustafa established the concept of b2-metric space in [2] this problem was solved.

The purpose of this paper is to introduce a weaker five-element function class than the literature [1] and to further proved that under the corresponding shrinking conditions, there is a unique fixed point for the mapping in the b_2-metric space.Thus, the conclusion of the literature [1] is generalized, and the fixed point theory of the metric space is also perfected.

II. Basic Definitions

Definition 1 A 2-metric space (X,d) is a set X with a non-negative real valued function $d: X\times X\times X\to[0,\infty)$ satisfying

i) for each distinct points $x,y\in X$,there exists an $u\in X$ such that $d(x,y,u)=0$;

ii) $d(x,y,z)=0$ if at least two of three points $x,y,z\in X$ are equal;

iii) $d(x,y,z)=d(x,z,y)=d(y,x,z)$ for all $x,y,z\in X$;

iv) $d(x,y,z)\le d(x,y,u)+d(x,u,z)+d(u.y.z)$ for all $x,y,z\in X$.

Definition 2 A b_2-metric space (X,d) is a set X with a non-negative real valued function $d: X\times X\times X\to[0,\infty)$ satisfying

i) for each distinct points $x,y\in X$,there exists an $u\in X$ such that $d(x,y,u)=0$;

ii) $d(x,y,z)=0$ if at least two of three points $x,y,z\in X$ are equal;

iii) $d(x,y,z)=d(x,z,y)=d(y,z,x)$ for all $x,y,z\in X$;

iv) for each $x,y,z,u\in X$, $s>1$, $d(x,y,z)\le s[d(x,y,u)+d(x,u,z)+d(u,y,z)]$.

Definition 3 A sequence $\{x_n\}$ in b_2-metric space (X,d) is called Cauchy sequence if for each $\varepsilon > 0$ and $a \in X$,there exists a positive integer N such that $n,m \geq \mathrm{N}$ implies $d(x_n, x_m, a) < \varepsilon$. The sequence is convergent to $x \in X$ and x is the limit of this sequenc(simply, $x_n \to x$)if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$.

Definition 4 A b_2-metric space is called complete if every Cauchy sequence converges.

Lemma 1 (Cauchy principle)[3] Let $\{x_n\}$ be a sequence with coefficient s and b_2-bounded number M in b_2-metric space. If for all $n,m \in N$ and $m > n$, $q > 0$ and $qs < 1$ such that

$d(x_n, x_{n+1}, x_m) \leq q^n M$

Then $\{x_n\}$ is a b_2-Cauchy Sequence.

Definition 5 Let φ be a five-element function which satisfies

$\varphi : (R^+)^5 \to R^+$

φ is continuous and monotonically increasing for each argument. At the same time,

$\varphi(t,t,t,2st,t) < t, \quad \forall t > 0$

s is the coefficient in b_2-metric space (X,d). And clearly $\varphi(0,0,0,0,0) = 0$.

III. Main results

Theorem Let (X,d) be a complete b_2-metric space and $T : X \to X$ satisfying

1. There is a $q \in (0,1)$ such that $qs < 1$ and

$$sd(x,y,a) \leq q\varphi(sd(Tx,Ty,a), sd(x,Tx,a), sd(y,Ty,a), sd(x,Ty,a), sd(Tx,y,a)) \qquad (1)$$

for all $x, y, a \in X$

2. If T is sequentially continuous, then T has an unique common fixed point on X .

Proof **R**andomly take $x_0 \in X$, we can construct the following sequence:

$x_{n-1} = Tx_n$

Then (1) becomes

$$\begin{aligned}
&sd(x_{n+1}, x_n, a) \\
&\leq q\varphi(sd(x_n, x_{n-1}, a), sd(x_{n+1}, x_n, a), sd(x_n, x_{n-1}, a), sd(x_{n+1}, x_{n-1}, a), sd(x_n, x_n, a)) \\
&\leq q\varphi(sd(x_n, x_{n-1}, a), sd(x_{n+1}, x_n, a), sd(x_n, x_{n-1}, a), sd(x_{n+1}, x_{n-1}, a), 0) \\
&\leq q\varphi(sd(x_n, x_{n-1}, a), sd(x_{n+1}, x_n, a), sd(x_n, x_{n-1}, a), s^2(d(x_n, x_{n-1}, a) + d(x_{n+1}, x_n, a) + d(x_{n+1}, x_{n-1}, a)), 0)
\end{aligned}$$

(2)

If $d(x_{n+1}, x_{n-1}, x_n) > 0$, then

$$\begin{aligned}
&sd(x_{n+!}, x_{n-1}, x_n) \\
&= sd(x_{n+1}, x_{n-1}, x_n) \\
&\leq q\varphi(sd(x_n, x_{n-1}, x_n), sd(x_{n+1}, x_n, x_n), sd(x_n, x_{n-1}, x_n), sd(x_{n+1}, x_{n-1}, x_n), 0) \\
&\leq q\varphi(0,0,0, sd(x_{n+1}, x_{n-1}, x_n), 0) \\
&\leq q\varphi(sd(x_{n+1}, x_{n-1}, x_n), sd(x_{n+1}, x_{n-1}, x_n), sd(x_{n+1}, x_{n-1}, x_n), 2s^2 d(x_{n+1}, x_{n-1}, x_n), sd(x_{n+1}, x_{n-1}, x_n)) \\
&\leq qsd(x_{n+1}, x_{n-1}, x_n)
\end{aligned}$$

This is a contradiction for that $0<q<1$, so there must be $d(x_{n+1},x_{n-1},x_n)=0$.
Then (2) becomes

$$sd(x_{n+1},x_n,a)$$
$$\le q\varphi(sd(x_n,x_{n-1},a),sd(x_{n+1},x_n,a),sd(x_n,x_{n-1},a),s^2(d(x_n,x_{n-1},a)+d(x_{n+1},x_n,a)),0)$$

(3)

If there is an $a\in X$ such that $d(x_n,x_{n-1},a)<d(x_{n+1},x_n,a)$, then (3) can be strengthened as

$$sd(x_{n+1},x_n,a)\le q\varphi(sd(x_{n+1},x_n,a),sd(x_{n+1},x_n,a),sd(x_{n+1},x_n,a),2s^2d(x_{n+1},x_n,a),sd(x_{n+1},x_n,a))$$
$$<qsd(x_{n+1},x_n,a)$$

This is also a contradiction for that $0<q<1$.

Then there must be $d(x_{n+1},x_n,a)\le d(x_n,x_{n-1},a)$ for all $a\in X$

In this case, strengthen (3) as

$$sd(x_{n+1},x_n,a)$$
$$\le q\varphi(sd(x_n,x_{n-1},a),sd(x_n,x_{n-1},a),sd(x_n,x_{n-1},a),2s^2d(x_n,x_{n-1},a),sd(x_n,x_{n-1},a))$$
$$\le qsd(x_n,x_{n-1},a)$$
$$\le q^nM$$

(4)

According to Lemma 1, $\{x_n\}$ is a Cauchy Sequence. Since X is complete, $\lim_{n\to\infty}x_n=u\in X$.

Next we prove that u is the unique common point of T

Let TX is a closed set.

Since $x_{n-1}=Tx_n\in TX$, there is a $z\in X$ such that $Tz=u$ when $n\to\infty$ for all $a\in X$.

According to shrinkage conditions we can conclude

If $d(x_{n+1},x_{n-1},x_n)>0$

$$sd(x_{n+1},x_{n-1},x_n)$$
$$=q\varphi(0,sd(z,Tz,a),0,sd(z,Tz,a),0)$$
$$\le q\varphi(sd(z,Tz,a),sd(z,Tz,a),sd(z,Tz,a),sd(z,Tz,a),sd(z,Tz,a))$$
$$<qsd(z,Tz,a)$$

It's obvious that this is a contradiction since $0<q<1$. So there must be $d(z,Tz,a)=0$ for all $a\in X$.

Thus $z=Tz=u$ which means u is the unique common point of T.

Next we need to prove that T has only one fixed point. We suppose that there is another point $z_0\in X$ such that $z_0=Tz_0=v$.

According to shrinkage conditions we can conclude

$$sd(z,z_0,a)$$
$$\le q\varphi(sd(Tz,Tz_0,a),sd(z,Tz,a),sd(z_0,Tz_0,a),sd(z,Tz_0,a),sd(Tz,z_0,a))$$
$$=q\varphi(sd(u,v,a),0,0,sd(u,v,a),sd(u,v,a))$$
$$<q\varphi(sd(u,v,a),sd(u,v,a),sd(u,v,a),2s^2d(u,v,a),sd(u,v,a))$$
$$<qsd(u,v,a)$$

This is also a contradiction for that $0<q<1$.

Therefore our assumption does not hold. So there must be $d(u,v,a)=0$ which means $u=v$.

So u is the unique common point of T.

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