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A Low Precision Quadrature Rule for Approximate Evaluation of Complex Cauchy Principal Value Integrals

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Abstract: *A four-point quadrature rule with degree of precision four for numerical integration of complex Cauchy principal value integrals has been constructed. An asymptotic error estimate and an error bound of the rule constructed have been derived. At the end, the rule constructed has been numerically verified.*

Keywords: *Cauchy Principal Value integrals, analytic function, quadrature rules, asymptotic error estimate, error bound and error constant. 2000 mathematical subject classification: No: 65D30.*

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I. Introduction

Cauchy Principal Value of a singular integral of the type:

$$I(f,c) = P\int_L \frac{f(z)}{z-c}dz; \tag{1.1}$$

where $f(z)$ is an analytic function in a simply connected domain Ω containig the line segment L from a to b in the complex plane C and c being the affix of a point on L is defined in Saff and Snider [16] as

$$I(f,c) = \lim_{\Delta\to 0}\left\{\int_{L_1}\frac{f(z)}{z-c}dz + \int_{L_2}\frac{f(z)}{z-c}dz\right\}; \tag{1.2}$$

(provided this limit exists) where L_1 and L_2 are line segments having end points a, c-Δ and c+Δ, b respectively on the path L.As far as it is known, the approximate evaluation of the integral given in (1.2) has not received sufficient attention by researchers associated with numerical integration of real Cauchy principal value integrals of the type

$$I(f,a) = P\int_{-1}^{1}\frac{f(x)}{x-a}dx; \qquad -1 < a < 1 \tag{1.3}$$

which exists if $f(x)$ is Hölder continuous in $-1 \le x \le 1$.

We find a collection of important but significant research works in the area of numerical integration of the integral (1.3) given in “Methods of numerical Integration” by Davis and Rabinowitz [11]. However, we do not find any mention of numerical integration of the integral (1.2) except the one due to Birkhoff-Young [4] for numerical integration of analytic function on a directed line segment in the complex plane C. The rule formulated by them is given by

$$\int_{z_0-h}^{z_0+h} f(z)dz \sim R_{BY}(f) = \frac{h}{15}\{24f(z_0) + 4[f(z_1)+f(z_3)] - [f(z_2)+f(z_4)]\} \tag{1.4}$$

where

$$z_k = z_0 + (i)^{k-1}h\,; \quad \text{for k = 1,2,3,4 and } i = \sqrt{-1}. \tag{1.5}$$

It is a five point quadrature rule of degree of precision five and its error term (E) satisfies

$$|E| \le \frac{1}{1890}|h|^7 \max_{z\epsilon S}\left|f^{(6)}(z_0)\right|, \tag{1.6}$$

where S denotes a square whose vertices are : $z_k = z_0 + (i)^k h\,;$ k=0,1,2,3 and $i = \sqrt{-1}$ [Ref. [11],pp.136].

The quadrature formula given in equation (1.4) with its error estimates given in equation (1.6) appears to be the first rule formulated by Birkhoff-Young [4] with error estimate (E) derived by Young [17] for the numerical integration of analytic function using two of the five nodes off the path of integration in the complex plane.

Later, in the year 1976, Lether [13] suggested to transfer the integral

$$I = \int_{z_0-h}^{z_0+h} f(z)dz; \tag{1.7}$$

to

$$I = h\int_{-1}^{1} f(z_0+ht)dt; \tag{1.8}$$

By using the transformation: $z = z_0 + ht$; $-1 \le t \le 1$ and then to numerically integrate the integral (1.8) by quadrature formula meant for approximation of real definite integrals in order to find an approximate value of the integral (1.7). It was also pointed out by Lether [13] that , the three point Gauss-Legender quadrature formula:

$$\int_{-1}^{1} f(x)dx \sim \frac{1}{9}\left[8f(0) + 5\left\{f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right\}\right]; \tag{1.9}$$

which is also of precision five integrates the integral (1.7) more accurately in comparison to the quadrature formula (1.4) with less number of function evaluation .

In the same vein in 1979, Acharya and Das [1] used the same transformation i.e. $z = z_0 + ht$; $-1 \le t \le 1$ to convert the Complex Cauchy Principal value integral of the type

$$I(f, z_0) = \int_{z_0-h}^{z_0+h} \frac{f(z)}{z-z_0} dz \tag{1.10}$$

to a real Cauchy Principal value integral

$$I(f, 0) = \int_{-1}^{1} \frac{hf(z_0+ht)}{t} dt \tag{1.11}$$

and then numerically integrated the integral (1.11) by the pair of rules formulated by Price [15] for approximation of real Cauchy Principal value integral of the type

$$I(f) = P\int_{-1}^{1} \frac{f(x)}{x} dx \tag{1.12}$$

to achieve approximate value of the Complex Cauchy Principal value integral given in equation (1.10).

Recently, Acharya and Mohapatra [2], Das and Hotta [7, 8, 9], Milovanovic [14] have formulated some quadrature formulas for numerical integration of the Complex Cauchy Principal value integral of the type given in equation (1.10).

Cauchy Principal value integrals quite often encountered by scientists and engineers in the studies of applied mathematics, theory of aerodynamics, scattering theory, crack problem in plane elasticity, the singular eigen function method in neutron transport and many other field of physical sciences. Since the evaluation of such integrals by analytic method in closed form is not possible in most of the situations, the approximation of these integral is inevitable.

Keeping in view the importance of approximate evaluation of Cauchy Principal value integrals (both real and complex) in pure and applied sciences, we desire to construct a quadrature formula for Complex Cauchy Principal value integral (1.10) with nodes used by Birkhoff and Young for the construction of quadrature formula for numerical integration of analytic function along a directed line segment in the complex plane. This formula also helps in the approximate evaluation of real Cauchy Principal value integrals of the type given in equation (1.12).

II. Formulation of the four-point Rule

Let the rule based on the nodes given in (1.5) be denoted by $R(f)$ and

$$R(f) = Af(z_0) + B\,[f(z_0+h) - f(z_0-h)] + C\,[f(z_0+ih) - f(z_0-ih)]. \tag{2.1}$$

The weights A,B,C are to be determined so that it exactly integrates polynomials of maximum degree. In other words,

$$I((z-z_0)^k) = R((z-z_0)^k) \quad \text{for k= 0,1,3.} \tag{2.2}$$

It is pertinent to note here that

$$I((z-z_0)^{2k}) = R((z-z_0)^{2k}); \quad \text{for k=1,2,3,...}$$

since the nodes associated with the proposed rule R(f) are symmetrically situated about the point z_0 .

Using the identities given in equation (2.2), the following set of three equations in the unknowns A, B, C is obtained:

$$\left.\begin{aligned} A &= 0, \\ B + iC &= 1, \\ \mathrm{B} - \mathrm{iC} &= \frac{-1}{3}. \end{aligned}\right\} \tag{2.3}$$

On solving the pair of linear equations in B and C we get

$$\mathrm{B} = \frac{2}{3} \quad \text{and} \quad \mathrm{C} = \frac{-i}{3} . \tag{2.4}$$

Thus, the quadrature rule proposed in the equation (2.1) is now given by

$$R(f) = \frac{2}{3}\;[f(z_0+h) - f(z_0-h)] - \frac{i}{3}[f(z_0+ih) - f(z_0-ih)] . \tag{2.5}$$

Degree of Precision of the rule $R(f)$

Let $$E(f) = I(f) - R(f) \tag{2.6}$$

Denote the truncation error in approximation of the Cauchy principal value of the integral $I(f)$ by the rule $R(f)$ given in equation (2.5). Now, it is easy to see that,

$$E((z - z_0)^k) = 0 \text{ ; k = 0(1)4}$$

and

$$E((z - z_0)^5) = -\frac{8}{5}h^5 \neq 0 . \tag{2.7}$$

So, the degree of precision of the rule given in (2.5) is four.

III. Error Analysis

We divide this section into two parts: in the first part, we derive an asymptotic error estimate and in the second part an error bound is obtained. In both the parts we assume that the function f(z) is analytic in a disc in the complex plane.

Asymptotic error estimate

Let f(z) be analytic in the disc

$$\Omega = \{z\epsilon C: |z - z_0| \leq \rho = r|h| \, ; r > 1\}$$

so that the points given in (1.5) are all interior to the disc Ω .

Using the Taylor's series expansion about $z = z_0$ we obtain

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{3.1}$$

Where

$$a_n = f^{(n)}(z_0)/(n)!$$

are the Taylor's coefficients.

Further, as the series given in equation (3.1) is absolutely and uniformly convergent in Ω, we get the following by integrating term by term the right hand side of (3.1).This yields

$$I(f) = \sum_{n=0}^{\infty} a_{2n+1} \frac{2h^{2n+1}}{2n+1} . \tag{3.2}$$

On the other hand, using the Taylor's series expansion for each term of $R(f)$ given in equation (2.5) about $z = z_0$ we obtain after simplification,

$$R(f) = 2hf^{'}(z_0) + \frac{1}{3}\frac{2h^3}{(3!)}f^{(3)}(z_0) + \frac{2h^5}{(5!)}f^{(5)}(z_0) + \cdots . \tag{3.3}$$

Then the truncation error $E(f)$ as defined in equation (2.6) is now given by

$$E(f) = -\frac{4}{5}\frac{2h^5}{(5!)}f^{(5)}(z_0) - \frac{4}{21}\frac{2h^7}{(7!)}f^{(7)}(z_0) - \cdots . \tag{3.4}$$

Thus from equation (3.4) we have the following:

Theorem – 1 :

If $f(z)$ is analytic in a certain domain Ω containing the line segment L then

$$E(f) = O(h^5). \tag{3.5}$$

for asymptotically small h.

Error Bound

The error bound of the truncation error $E(f)$ for the rule given in equation (2.1) is obtained by using the technique due to Lether[12] for fully symmetric quadrature rules and it is given in the following theorem.

Theorem - 2:

Let the function $f(z)$ is analytic in the disc Ω. Then the upper bound for the truncation error $E(f)$ is given by

$$|E(f)| \leq 2\, M(\rho)\, e(r) \tag{3.6}$$

where

$$M(\rho) = max_{|z - z_0| = \rho} |f(z)|$$

and

$$e(r) = \left|\ln\left(\frac{r+1}{r-1}\right) - \left(\frac{2r(3r^2+1)}{3(r^4-1)}\right)\right| \tag{3.7}$$

which tends to zero as r → ∞.

Proof:

Expanding $f(z)$ in Taylor's series about $z = z_0$ and using the transformation $z = z_0 + ht,\ t\epsilon[-1,1]$ we obtain,

$$E(f) = \sum_{\mu=2}^{\infty} a_{2\mu+1} h^{2\mu+1} E(t^{2\mu+1}) . \tag{3.8}$$

Now using Cauchy's inequality for derivatives [5] in equation (3.8) we arrive at

$$|E(f)| \leq 2\mathrm{M}\,(\rho)\ \sum_{\mu=2}^{\infty} \frac{1}{r^{2\mu+1}}\, |E(t^{2\mu+1})|. \qquad (3.9)$$

But $$E(t^{2\mu+1}) = 2\left[\frac{1}{2\mu+1} - \frac{1}{3}\{2 + (-1)^{\mu}\}\right] < 0; \quad \mu \geq 2.$$

Therefore, as it is done by Lether [12] , we can rewrite the equation (3.9) as

$$|E(f)| \leq 2\ \mathrm{M}(\rho)\ e(r)$$

where $$e(r) = \left|E\left(\left(1 - \frac{t}{r}\right)^{-1}\right)\right|.$$

It is not difficult to derive

$$E\left(\left(1 - \frac{t}{r}\right)^{-1}\right) = \ln\left(\frac{r+1}{r-1}\right) - \left(\frac{2r(3r^2+1)}{3(r^4-1)}\right)$$

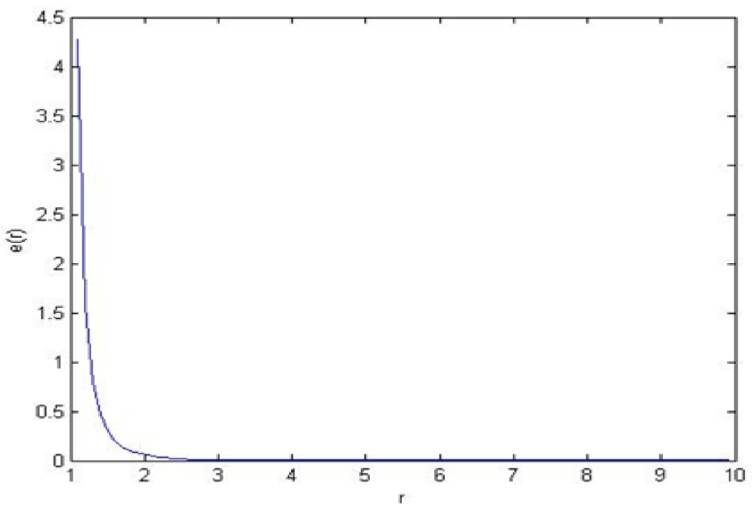
by applying the quadrature rule (2.5) to the function $\emptyset(t) = (1 - \frac{t}{r})^{-1}$; $r > 1$.

Analytically e(r) → 0 as r → ∞ ,but practically e(r) = 0 for r =43.7 which is shown below in **Table – 1** of error constant for different values of r (> 1) **.**Now the graph of r verses e(r) is given in **figure –** 1 and from this it is observed that e(r) approaches to zero as r → ∞.

Table-1

r	e(r)
1.6	**0.20081**
1.8	**0.101684**
2.1	**0.0438025**
2.7	**0.0117615**
4.3	**0.0011059**
6.7	**0.0001192**
10.6	**0.0000119**
17.1	**0.0000011**
27.5	**0.0000001**
43.7	**0.0000000**

Figure-1



IV. Numerical Verification

The integrals

$$I_1 = P \int_{-0.1i}^{0.1i} \frac{e^z}{z}\,dz$$

and

$$I_2 = P \int_{-0.1}^{0.1} \frac{e^x}{x}\,dx$$

have been numerically integrated by the four-point degree four rule $R(f)$ given in equation (2.5) and the results of numerical integration of these CPV integrals are given in Table-2

Table-2

Integral	Approximate Value	Exact Value	Absolute Error
I_1	$0.1998891i$	$0.1998889i$	2.0×10^{-7}
I_2	0.2001113	0.2001111	2.0×10^{-7}

V. Conclusion

1. The rule does not require the evaluation of function at $z = z_0$: the point of singularity. Also, it is not required to evaluate the derivative of the integrand at any of its nodes, which is a positive advantage over the existing rules where evaluation of derivatives is required at $z = z_0$.
2. We have noted that as the range of integration increases the accuracy of approximation gradually decreases and this is obvious since the rule given in (2.1) is a low precision rule i.e. four. Such a rule will be useful in case the higher order derivatives of the integrand (order greater than five) do not exist.
3. This rule may also be employed in adaptive integration of analytic functions over a line segment; the point of singularity may be indentated by an interval $(-\varepsilon, \varepsilon)$; $\varepsilon > 0$ in which this rule for sufficiently small ε may be applied and on the rest of the intervals suitable quadrature rules meant for integration of analytic functions may be used.

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