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Direct Product of Brandt Semigroup and Its Rank as a Class of Algebra

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Abstract: *The rank of direct product of Brant SemigroupB(G, n) where n*≥ 2 *and* $|G| \geq 1$, *is presented as a class of Algebra in comparison to the sum. This study is also extended to the computation of the rank of direct product of cancellativesemigroups.*

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I. Introduction and Preliminaries

The notion of rank of an algebraic system originated from linear algebra in connection with determination of cardinality of the maximal linearly independent subset of a vector space or as the cardinality of minimal generating set of a given vector space. In linear algebra the cardinality of maximal linearly independent subset of a vector space is the same as the cardinality of minimal generating set for the vector space. Each of these defines the rank of the vector space V. With the development of the concept of independence in general algebraic system (Marczewski, [8]), the notion of rank became extended to general algebraic systems, such as semigroups, groups, etc. While in linear algebra the rank of a vector space V as the cardinality of maximal linearly independent subset of V is the same as the cardinality of generating set of V, in general algebraic systems this is not the case. Indeed it is well known that the rank of an element A as a minimal cardinality of the generating set of A is not the same as the rank of A as the maximal cardinality of independent subset of A (Howie [6]).

Many investigations have been done on various concepts of ranks in algebraic systems including algebraic classes of semigroups. Gomes [4], Howie [6], Ribeiro [9], Gould [5], Cegarra [2] have determined ranks of various types of semigroups. Most of these investigations have been applied to certain finite semigroups. For example, Howie [6] investigated the properties of rank of a finite semigroups, while Cegarra [2] characterized rank of commutative cancellativesemigroup in terms of embeddability into a rational vector space of the greatest power cancellative image. Determining the rank of different algebras in a given class may be a variety or a quasivariety

We will stick specifically to the notation and terminologies by Howie [7] for semigroups, and for classes of algebras Burris and Sankappanavar [1]. Where necessary we also use Clifford and Preston [3]. Brandt semigroupand Cancellativesemigroup is discussed in section 3 and 4 respectively. A typical class of algebraic system contains for any algebraic system, the subalgebra, direct product in particular subdirect product of its algebraic system. Depending on whether the class of algebra is a variety or quasivariety, such a class must contain homomorphic copies of its algebra in the case of the former or does not contain copies in the case of the later (quasivariety).

A typical approach to determining the rank of a semigroup or indeed any algebraic system normally requires;

1. Determining possibly all subsets of the semigroup S which are independent.
2. Determine which of them are independent subset of S which generate S.
3. From these sets get the one with maximum cardinality which generates S.
4. Also obtain the one with minimum cardinality which generates the S
5. Find the independent subset with cardinality between them that generates S.
6. From the independent subset of the maximum cardinality which generates S, we have upper rank.
7. One can also find the minimum cardinality of every subset which generate the semigroup$S$ and it is called the large rank.

The concept stated above in (1) to (7) has been widely used by many authors. It offers a precise approach to determination of ranks.

Rees Matrix: Let G be a group with identity element e, and let I, Λ be nonempty sets. Let $P = (P_{\lambda i})_{i \in I, \lambda \in \Lambda}$ be a $\Lambda \times I$ matrix with entries in the 0-group $G^0 = G \cup \{0\}$, and suppose that P is regular, in the sense that no row or column of P consists entirely of zeros. Formally,

$$(for\ all\ i \in I, there\ exist\ \lambda \in \Lambda)\ such\ that\ P_{\lambda i} \neq 0$$

$$(for\ all\ \lambda \in \Lambda\ , there\ exist\ i \in I)\ such\ that\ P_{\lambda i} \neq 0$$

Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a composition on S by

$$(i,a,\lambda)(j,b,\mu) = \begin{cases} i, aP_{\lambda i}b, \mu) & if\ P_{\lambda j} \neq 0 \\ 0 & if\ P_{\lambda j} = 0 \end{cases}$$

$$(i,a,\lambda)0 = 0(i,a,\lambda) = 00 = 0$$

Then S is a semigroup. This semigroup is called a Rees Matrix semigroup, denoted by $M^0[G; I, \Lambda; P]$. That is the $I \times \Lambda$ Rees matrix semigroup over the 0-group G^0 with the regular sandwich matrix P.

The semigroup$S = (I \times G \times \Lambda) \cup \{0\}$ above is a completely 0-simple semigroup.

Let $S = M^0[G; I, \Lambda; P]$, be a completely 0-simple inverse semigroup. Since each $\mathcal{L}$-class and each $\mathcal{R}$-class contains exactly one idempotent, there is exactly one non-zero entry in each row and each column of the sandwich matrix P. There is thus a bijection from I onto Λ define by the rule that $i \rightarrowtail \lambda$ if and only if $P_{\lambda i} \neq 0$. Hence |Λ|= |I|, and we may suppose that Λ and I are ordered so that the non-zero entries occur on the main diagonal. Since I and Λ are index sets we may in effect suppose that Λ = I. Thus $S = M^0[G; I, I; P]$, where P is a diagonal matrix.

Now let Δ = [δ_{ij}] be the $I \times I$ matrix given by

$$\delta_{ij} = \begin{cases} e & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

By theorem3.4.1 (Howie 1995), $S = M^0[G; I, I; \Delta]$, since there are elements $u_i = P_{ii},\ v_j = e\ (i, j \in I)$ such that $P_{ij} = v_j \delta_{ij} u_i$ for all $i, j\ in\ I$.

The semigroup$M^0[G; I, I; \Delta]$ is called a Brandt semigroup. If I is a finite set, with |I| =n, we usually denote the Brandt by $B(G, n)$. We particularly write $B(\{e\}, n)$, a semigroup of order $2n + 1$, as B_n

Direct Product of algebra: The direct product of the family of algebra is denoted by $\pi\mathfrak{E}_i$ and can be viewed as n-tuples $(a_1, a_2, \cdots, a_n)$where $a_i \in \mathfrak{E}_i$. The direct product $\pi\mathfrak{E}_i$ of an algebra $\mathfrak{E}_i$, i∈I is an algebra. $\mathfrak{E} = \langle \pi A_i, F \rangle$where$\pi A_i$ is the direct product of base set and F operation are n-ary operation on polynomials in πA_i. If $\mathfrak{E}$=$\pi\mathfrak{E}_i$ where $\mathfrak{E}_i \cong \mathfrak{E}_1 = \cdots = \mathfrak{E}_n$ then $\mathfrak{E}$=$\pi_{i=1}^n \mathfrak{E}_i$ is the n-th direct power of $\mathfrak{E}_i'$s. Without specifying the number of I, we simply write $\pi\mathfrak{E}_{i \in I}$. Let $\mathfrak{E}$=$\pi\mathfrak{E}_i$ be the direct product of $\mathfrak{E}_i, i \in I$. The subalgebra of $\mathfrak{E}$ is a subdirect product of algebra $\mathfrak{E}_i$.

Variety: A nonempty class K of algebras of type τ is called a variety if it is closed under subalgebras, homomorphic images, and direct products

1. Facts.

II. Theorem [10]

1. Let S and T be two infinite semigroups. Then S x T is finitely generated if and only if both S and T are finitely generated and $S^2 = S$and $T^2 = T$.

Proof: Denote by $\pi_s: S \times T \rightarrow S$ the natural projection. If A is a generating set for $S \times T$ then $\pi_s(A)$ is a generating set for S. In particular, if $S \times T$ is finitely generated then so is S, since π_s is a onto map (epimorphism).

Assume that $S^2 \neq S$, so that S has an indecomposable element s. But then each of the infinitely many elements $(s,t)(t \in T)$ is indecomposable in $S \times T$, and hence it belongs to every generating set for $S \times T$, contradicting the assumption that $S \times T$ is finitely generated.

For decomposable semigroupwhere $S^2 = S$, let

$$A = \{a_i : i \in I\}$$

be a generating set for S. Then there exist elements $s_i \in S\ (i \in I)$ and a mapping $\zeta: I \rightarrow I$ such that $a_i = a_{\zeta(i)} s_i$ for all $i \in I$

Each a_i can be written as a product $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ of generators with $k \geq 2$. Now define $\zeta(i) = i_1$ and $s_i = a_{i_1}, a_{i_2}, \cdots, a_{i_k}$

2. Let S and T be two semigroups satisfying $S^2 = S$ and $T^2 = T$

Let $A = \{a_i : i \in I\}$ and $B = \{b_j : j \in J\}$ be generating sets for S and T respectively.

Choose elements $s_i \in S\ (i \in I)$ and $t_j \in T\ (j \in J)$ and functions

$\zeta: I \rightarrow I$and$\theta: J \rightarrow J$ so that $a_i = a_{\zeta(i)} s_i$ for all $i \in I$ and $b_j = b_{\theta(j)} t_j$ for all $j \in J$. Then the set

$$(A\ U\ \{s_i : i\ \in I\})\ \times\ (B\ U\ \{t_j : j\ \in\ J\})$$

generates S x T.

Proof: Let s ∈ S be arbitrary, and assume that s can be decomposed into a product of m generators from A. By successively replacing an arbitrary generator a_i by the product $a_{\zeta(i)} s_i$, we see that for every n ≥ m the element s can be expressed as a product of n elements from $A\ U\ \{s_i : i\ \in I\}$. Similarly, if an element t ∈ T can be expressed

as a product of m generators from B, then for every n ≥ m it can be expressed as a product of n elements from $B\ U\ \{t_j: j\ \in\ J\}$.
Now let s ∈ S and t ∈ T be arbitrary. Assume that s can be written as a product of m generators from A, and that t can be written as a product of n generators from B. Let p = max(m, n), and write s and t as products

$$s = \alpha_1\alpha_2 \cdots \alpha_p$$
$$t = \beta_1\beta_2 \cdots \beta_p$$

of p elements from $A\ U\ \{s_i: i\ \in I\}\ and\ B\ U\ \{t_j: j\ \in\ J\}$, respectively. Now we can write (s, t) as a product of elements from

$$(A\ U\ \{s_i: i\ \in I\})\ \times\ (B\ U\ \{t_j: j\ \in\ J\})$$

as follows:

$$(s,t) = (\alpha_1, \beta_1)(\alpha_2, \beta_2) \cdots (\alpha_p, \beta_p)$$

as required.
Remark
If S and T have the property that $S^2 = S$and $T^2 = T$, then we also have $(S \times T)^2 = S \times T$. The converse is also true: if $(S \times T)^2 = S \times T$ then $S^2 = S$and $T^2 = T$, since S and T are homomorphic images of S x T. This observation enables one to generalise the above Theorem 1, to arbitrary finite direct products $S_1 \times S_2 \times \cdots \times\ S_k$ of infinite semigroups. This direct product is finitely generated if and only if each S_i $(1\ \leq i \leq k)$ is finitely generated and satisfies $S_i^2 = S_i$.

III. Brandt Semigroup

3.**1 Proposition**: The direct product of a Brandt semigroup is a Brandt semigroup.
Proof
Let $B(G_1, n)$ and $B(G_2, m)$ be two different Brandt semigroups
$B(G_1, n) = (\{1, 2, \ldots, n\} \times G_1 \times \{1, 2, \ldots, n\}) \cup \{0\} = (i, a, j)\ \ wherei, j$are index set and $a \in G_1$
Similarly,

$$B(G_2, n) = (\{1, 2, \ldots, m\} \times G_2 \times \{1, 2, \ldots, m\}) \cup \{0\} = (i, b, j)\ \ for\ b \in G_2$$

$B(G_1, n) \times B(G_2, m) = \{(U, V): U \in B(G_1, n)andV \in B(G_2, m)$
We denote $B(G_1, n) \times B(G_2, m)$by $B(G,\ n^{'})\ Where\ a, b \in G$. We now show that this product is associative.
$(u_1, v_1)\big((u_2, v_2)(u_3, v_3)\big) = ((u_1, v_1)(u_2, v_2))(u_3, v_3)$where$u_1 \in U,\ \ v_1 \in V, for\ all\ i, j$ 3.6.1

$$(u_1, v_1)\big((u_2, v_2)(u_3, v_3)\big) = (u_1u_2u_3, v_1v_2v_3) \qquad 3.6.2$$
$$\big((u_1, v_1)(u_2, v_2)\big)(u_3, v_3) = (u_1u_2u_3, v_1v_2v_3) \qquad 3.6.3$$

Hence (3.6.1) is satisfied
Recall that $U \in B(G_1, n)$ and $V \in B(G_2, m)$
$\Rightarrow u_1 = (i, a_1, j), u_2 = (i, a_2, j)\ \ and\ \ u_3 = (i, a_3, j),\ \ \ v_1 = (i, b_1, j), v_2 = (i, b_2, j)\ \ and\ \ v_3 = (i, b_3, j)$
From (3.6.1) we have that
$B(G,\ n^{'}) = \{((i, a_1, j)(i, a_2, j)(i, a_3, j)),\ \ ((i, b_1, j)(i, b_2, j)(i, b_3, j)).$
The multiplication is done on the rule governing Brandt semigroup. That is

$$(i, a, j)(k, b, l) = \begin{cases} (i, ab, l) & if\ j = k \\ 0 & otherwise \end{cases}$$

3.2 Example

Consider the five element Brandt Semigroup,

$$B_2 = \{a, b, ab, ba, o \mid aba = a, bab = b, a^2 = b^2 = 0\}$$

The minimal generating subset of B_2 is $\{a, b\}$ and rank (B_2)=2. The product of $B_2 \times B_2 \times B_2$ in example (1) above is
$\{a, b, ab, ba, o\} \times \{a, b, ab, ba, o\} \times \{a, b, ab, ba, o\}$
=$\{(a, a), (a, b), (a, ab), (a, ba), (a, 0), (b, a), (b, b), (b, ab), (b, ba), (b, 0), (ab, a),$
ab, b, ab, ab, ab, ba, ab, 0, ba, a, ba, b, ba, ab, ba, ba, ba, 0, 0, a, 0, b, 0, ab, 0, ba, 0, 0×a, b, ab, ba, o =

$\{\big((a, a), a\big), \big((a, a), b\big), \big((a, a), ab\big), \big((a, a), ba\big), \big((a, a), 0\big), \big((a, b), a\big), \big((a, b), b\big), \big((a, b), ab\big),$
$\big((a, b), ba\big), \big((a, b), 0\big), \big((a, ab), a\big), \big((a, ab), b\big), \big((a, ab), ab\big), \big((a, ab), ba\big), \big((a, ab), 0\big), \big((a, ba), a\big),$
$\big((a, ba), b\big), \big((a, ba), ab\big), \big((a, ba), ba\big), \big((a, ba), 0\big), \big((a, 0), a\big), \big((a, 0), b\big), \big((a, 0), ab\big), \big((a, 0), ba\big),$

$\big((a, 0), 0\big), \cdots, ((0, 0), 0)\}$
Order of $(B_2 \times B_2 \times B_2)$ =125.
Minimal generator = $\{\big((a, a), a\big), \big((b, b), b\big)\}$ which has 2 as the rank.
Rank $(B_2 \times B_2 \times B_2)$=2 ≤ Rank (B_2)+ Rank (B_2)+ Rank (B_2)< 2+2+2=6

3.3 Corollary.
Rank $(B_2 \times B_2 \times B_2) \leq$ Rank (B_2)+ Rank (B_2)+ Rank (B_2)
Proof: This straightforward from the above example.

4.1 Cancellativesemigroup

If $S_1 S_2, S_3$are cancellative then
$T = S_1\ X\ S_2\ ,\ S_3$are cancellative. Also, $S_1,\ S_2 \times S_3$ are cancellative, so is $S_1\ \times\ S_2 \times S_3$ cancellative.
$(S_1\ \times\ S_2)Q \cong S_1 Q\ \times\ S_2 Q$
rank$(S_1 \times S_2) = rank(S_1 Q\ \times\ S_2 Q) = rank\ S_1 Q +\ rank\ S_2 Q.$
rank$(S_1\ \times\ S_2\ \times\ S_3) = rank\ (S_1 Q\ \times\ S_2 Q\ \times\ \ S_3 Q)$
$= rank\ S_1 Q +\ rank\ S_2 Q +\ rank\ S_3 Q = rank\ S_1 +\ rank\ S_2 +\ rank\ S_3.$
For $S_1\ ,\ S_2\ ,\ S_3, \cdots, S_k$ cancellative
rank$(S_1\ \times\ S_2\ \times S_3, \ldots, S_k) =$ rank $(S_1 Q\ \times\ S_2 Q\ \times \cdots \times\ S_k Q)$
$= rank\ S_1 Q + rank\ S_2 Q + \cdots + rank\ S_K Q$
$= rank\ S_1 + rank\ S_2 + \cdots + rank\ S_K$
Let $T = (S_1\ \times\ S_2\ \times\ \cdots \times S_k)$
$T\ \times\ S_{k+1} = S_1\ \times\ S_2\ \times \cdots \times S_k \times\ S_{k+1}.$
$rank\ ((S_1\ \times\ S_2\ \times\ \ldots \times\ S_k)\ \times\ S_{k+1}\ )$
$= rank\ (S_1\ \times\ S_2\ \times\ \ldots \times\ S_k)\ Q\ \times\ S_{k+1} Q$
$= rank\ (\ S_1 Q\ \times\ S_2 Q\ \times \ldots \times S_k Q) + rank\ S_{k+1} Q$
$= rank\ \ S_1 Q + rank\ S_2 Q +\ \ldots + S_k Q + rank\ S_{k+1} Q$
$= rank\ \ S_1 + rank\ S_2 + \cdots + rank\ S_k + rank\ S_{k+1}$

4.2. Corollary

If $S_1 S_2, S_3$are cancellative, then
rank$(S_1\ \times\ S_2\ \times S_3 \times \ldots \times S_k) = rank\ S_1 + rank\ S_2 + \cdots + rank\ S_K$

V. Conclusion

The rank of the direct product of any algebraic system depends on the structure of the system. This rank may be equal to or greater than the rank of the sum of each as seen in our theorem and corollaries.

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