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On I and $I^{\mathcal{K}}$-convergence in n-normed linear Spaces

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Abstract: *In this paper, we introduce the concept of I-convergence, $I^{\mathcal{K}}$- convergence, I-cluster points and I-limit points in a n-normed linear space and thereby we prove some of its basic properties of I-convergence and $I^{\mathcal{K}}$-convergence on such spaces.*

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I. Introduction

The notion of statistical convergence was introduced first by Fast ([3]). Subsequently Kostyrko et al.([10]) extended this idea to the concept of I-convergence of sequences in a metric space with the notion of an ideal of the set of positive integers . Infact the notion of I-convergence is a generalization of statistical convergence and it also provides a general outline to study the properties in respect of various types of convergence.Taking into consideration of such notion of I-convergence much work had been done in different forms of convergences via I-cluster points, I-limit superior, I-limit inferior(see [1],[2],[5],[6],[11])in different topological structured spaces. Based on the concept of 2-metric spaces and 2-normed linear space introduced by S.Gähler (see [12]-[13]) a study on n-norm theory led by Gunawan and Mashadi (see [4]) gave into the development of a n-normed space which is a generalization of 2-normed space. Further the investigations on ideal convergence and $I^{\mathcal{K}}$- convergence of a sequence in a 2-normed linear space was done by M.Gürdal (see [7]) and Madjid Eshaghi Gordji et al (see [9]) respectively. Also the work on ideal convergence of a sequence in n-normed spaces could also be found in Gürdal and Sahiner (see [8]). In a natural way, one may invite these concepts of convergence for its general study on such n-normed linear spaces and thereby we have been able to prove here some of its properties on convergence of a sequences. Through out the paper $\mathbb{N}$ denotes the set of positive integers.

II. Preliminaries

Definition 2.1 [4]. Let X be the linear space . For $n \in \mathbb{N}$, let $(\|.,\ldots,.\|)$ be a non -negative real valued function on $X \times X \times ... \times X = X^n$ satisfying the following conditions:
(i) $\| x_1, x_2, \ldots, x_n \| = 0$ if and only if $x_1, x_2, \ldots, x_n \in X$ are linearly dependent.
(ii) $\| x_1, x_2, \ldots, x_n \|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n \in X$.
(iii) $\| x_1, x_2, \ldots, \alpha x_n \| = |\alpha| \, \| x_1, x_2, \ldots, x_n \|$,where $\alpha \in \mathbb{R}, x_1, x_2, \ldots, x_n \in X$.
(iv) $\| x_1, x_2, \ldots, x_{n-1}, y + z \| \le \| x_1, x_2, \ldots, x_{n-1}, y \| + \| x_1, x_2, \ldots, x_{n-1}, z \|$, for all $y, z, x_1, x_2, \ldots, x_{n-1} \in X$
Then $\|.,\ldots,.\|$ is called a n-norm on X and the corresponding pair $(X, \|.,\ldots,.\|)$ is called a n-normed linear space.

Example 2.2 [4]. The space $X = \mathbb{R}^n$ is equipped with the following n-norm:

$$\| x_1, x_2, \ldots, x_n \| = \left|\det \begin{pmatrix} x_{11}\, x_{12} \ldots\ldots\ldots x_{1n} \\ x_{21}\, x_{22} \ldots\ldots\ldots x_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n1}\, x_{n2} \ldots\ldots\ldots x_{nn} \end{pmatrix}\right|$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$,for each $i = 1,2,\ldots,n$.

Definition 2.3[4]. A sequence(x_n) in a n-normed linear space $(X, \|.,\ldots,.\|)$is said to be a Cauchy if $\lim_{k,m\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_k - x_m \| = 0$, for all $z_1, z_2, \ldots, z_{n-1}$ in X.

Definition 2.4[4]**.** A sequence (x_n) in a n-normed linear space$(X, \|.,\ldots,.\|)$is said to be convergent if there is a point x in X such that $\lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_k - x \| = 0$, for all $z_1, z_2, \ldots, z_{n-1}$ in X. If (x_n) converges to x,we write $x_n \to x$ as $n \to \infty$.

Definition 2.5 [4**].** A n-normed linear space in which every Cauchy sequence in X is convergent to an element of X is called a n-Banach space.

Definition 2.6 [9**].** A nonempty family $I \subseteq \mathcal{P}(Y)$ of subset of a nonempty set Y is said to be an ideal in Y if:
(i) $\phi \in I$
(ii) $A, B \in I$ implies $A \cup B \in I$
(iii) $A \in I, B \subseteq A$ implies $B \in I$.
I is called a proper ideal if $Y \notin I$ and I is not a proper ideal if $I = \mathcal{P}(Y)$. The ideal of all finite subsets of a given set Y is called Fin.

Definition 2.7. An ideal $I \subseteq \mathcal{P}(Y)$ is said to be non-trivial if $I \neq \phi$ and $Y \notin I$.

Definition 2.8 [9**].** A non-trivial ideal I in Y is said to be admissible if $\{x\} \in I$ for each $x \in Y$.

Definition 2.9 [9**].** A nonempty family $\mathcal{F} \subseteq \mathcal{P}(Y)$of subset of a nonempty set Y is said to be a filter in Y if:
(i) $\phi \in \mathcal{F}$
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$
(iii) $A \in \mathcal{F}, A \subseteq B$ implies $B \in \mathcal{F}$
If I is non-trivial ideal in Y, $Y \neq \phi$, then the class $F(I) = \{M \subset Y : (\exists A \in I) M = Y - A\}$ is a filter on Y, called the filter associated with Y.

Definition 2.10[3]. Let E be a subset of natural numbers $\mathbb{N}$ and $j \in \mathbb{N}$.The quotient
$d_j(E) = \text{card}(E \cap \{1, \ldots, j\})/j$ is called the jth partial density of E where $d_j$is a probability measure on $\mathcal{P}(N)$ with support $\{1, \ldots, j\}$. The limit $d(E) = \lim_{j\to\infty} d_j(E)$is called the natural density of $E \subseteq \mathbb{N}$ (if exists).Clearly,finite subsets have natural density zero and $d(E^c) = 1 - d(E)$ where $E^c = \mathbb{N} - E$, i.e. the complement of E.

Definition 2.11. A sequence (x_n) of elements in a n-normed linear space X is said to be statistically convergent to $x \in X$ if for each $\varepsilon > 0$ and for non zero $z_1, z_2, \ldots, z_{n-1}$ in X the set
$A(\varepsilon) = \{n \in \mathbb{N} : \| z_1, z_2, \ldots, z_{n-1}, x_n - x \| \geq \varepsilon\}$ has natural density zero.In other words for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \text{card}(\{n \geq k : \| z_1, z_2, \ldots, z_{n-1}, x_k - x \| \geq \varepsilon\}) = 0$$

Definition 2.12[8]**.** Let $I \subseteq \mathcal{P}(\mathbb{N})$ be a nontrivial ideal in $\mathbb{N}$. The sequence (x_n) of X is said to be I-convergent to $x \in X$ if for each $\varepsilon > 0$ and non zero $z_1, z_2, \ldots, z_{n-1}$in X, the set

$$A(\varepsilon) = \{k \in \mathbb{N} : \| z_1, z_2, \ldots, z_{n-1}, x_k - x \| \geq \varepsilon\} \in I$$

If (x_n) is I-convergent to $x \in X$ then we write $I - \lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_k - x \| = 0$; or $I - \lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_k \| = \| z_1, z_2, \ldots, z_{n-1}, x \|$. The number $x \in X$ is called the I-limit of the sequence (x_n).

Further we can see some examples of ideals and its corresponding I-convergence [see 14]. It is immediate that the following holds.
(I) Let I_f be the family of all finite subsets of $\mathbb{N}$. Then I_f is an admissible ideal in $\mathbb{N}$ and I_f-convergence of a sequence in a n-normed linear space X coincides with its usual convergence in X.
(II) Put $I_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Then I_d is an admissible ideal in $\mathbb{N}$ and I_d-convergence of a sequence in a n-normed linear space coincides with its statistical convergence in X.

Remark 2.13[8]. If I is an admissible ideal in n-normed linear space $(X, \|.,\ldots,.\|)$ then the convergence of a sequence in X implies its I-convergence in X.

We are now in a position to note that which of the following holds for the convergence of a sequence in X implies its I-convergence in X.
(A) Every constant sequence $(x, x, \ldots., x, \ldots, .)$ converges to x in a n-normed linear space.
(B) The limit of any convergent sequence in a n-normed linear space X is uniquely determined.
(C) If a sequence (x_n) in X has a limit x in X, then each of its subsequence has the same limit.
(D) If each subsequence of the sequence (x_n) in X has a subsequence which converges to x in X, then (x_n) converges to x in X.

Proposition 2.14 [8]. Suppose that X is a n-normed linear space having at least two points. Let $I \subset \mathcal{P}(Y)$ be an admissible ideal, then
(i) The I -convergence in X satisfies (A),(B) and (D).
(ii) If I contains an infinite set, then I-convergence in X does not satisfy (C)

Example 2.15 [8]. Let $I = I_d$. Define a sequence (x_n) in a n-normed linear space $(X, \|., \ldots, .\|)$ by

$$x_n = \begin{cases} (0,0, \ldots., n) & \text{if } n = i^2, i \in \mathbb{N} \\ (0,0, \ldots\ldots, 0) & \text{otherwise} \end{cases}$$

Let $x = (0,0, \ldots, 0) \in X$. Then $I - \lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_k \| = \| z_1, z_2, \ldots, z_{n-1}, x \|$. But the sequence (x_n) is not convergent to x.

We now conclude the fact that I-limit operation for the sequence in n-normed linear space $(X, \|., \ldots, .\|)$ is linear with respect to summation and scalar multiplication.

Theorem 2.16[8]. Let I be an admissible ideal in a n-normed linear space X. For each $z_1, z_2, \ldots., z_{n-1}$ in X, if $I - \lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_k - x \| = 0$ and $I - \lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, y_k - y \| = 0$ then
(i) $I - \lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, (x_k + y_k) - (x + y) \| = 0$ and
(ii) $I - \lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, c(x_k - x) \| = 0$ for all $c \in \mathbb{R}$

III. Main Results

Note that there is a strong connection between statistical cluster points and ordinary limit points of a given sequence. We will prove that analouge fact is also satisfied for I-cluster points and I-limit points for a given sequences in a n-normed linear space X.

Definition 3.1. Let $I \subset \mathcal{P}(X)$be an admissible ideal in X and $x = (x_n)$ be a sequence in X. Then
(i) A number $\xi \in X$ is said to be an I-limit point of x if there is a set $M = \{m_1 < m_2 < \ldots m_{k-1}\} \subset N$ such that $M \notin I$ and $\lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_{m_k} - \xi \| = 0$ for each non zero $z_1, z_2, \ldots., z_{n-1}$ in X. The set of all I-limit point of x is denoted by $I(\wedge_x^n)$.
(ii) A number $\xi \in X$ is said to be an I -cluster points of x if for each $\varepsilon > 0$ the set $\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - \xi \| < \varepsilon\} \notin I$ for each non zero $z_1, z_2, \ldots., z_{n-1}$ in X. The set of all I-cluster points of x is denoted by $I(\Gamma_x^n)$.

Theorem 3.2. Let $I \subset \mathcal{P}(X)$ be an admissible ideal in X. Then for each sequence $x = (x_n)$ in a n-normed linear space X we have $I(\wedge_x^n) \subset I(\Gamma_x^n)$ and the set $I(\Gamma_x^n)$ is a closed set.

Proof. Let $\xi \in I(\wedge_x^n)$.
Then there exist a set $M = \{m_1 < m_2 < \cdots\} \notin I$ such that $\lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_{m_k} - \xi \| = 0$ for each non zero $z_1, z_2, \ldots., z_{n-1}$ in X. Thus for each $\delta > 0$ there exist $k_0 \in \mathbb{N}$ such that for $k > k_0$ and each nonzero $z_1, z_2, \ldots., z_{n-1}$ in X, we have by (3.1)

$$\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - \xi \| < \delta\} \supset M \backslash \{m_1, m_2, \ldots, m_{k_0}\}$$

and so $\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - \xi \| < \delta\} \notin I$. Therefore $\xi \in I(\Gamma_x^n)$.

Let $y \in \overline{I(\Gamma_x^n)}$. For $\varepsilon > 0$. So there exists a $\xi_0 \in X$ such that $\xi_0 \in I(\Gamma_x^n) \cap B_u(y, \varepsilon)$. Choose $\delta > 0$ such that $B_u(\xi_0, \delta) \subset B_u(y, \varepsilon)$. Therefore we have

$$\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - y \| < \varepsilon\} \supset \{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - \xi_0 \| < \delta\}$$

Hence $\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - y \| < \varepsilon\} \notin I$ which in turn implies that $y \in I(\Gamma_x^n)$.

Definition 3.3. Let $I \subset \mathcal{P}(X)$ be an admissible ideal in X and let $x = (x_n)$ be a sequence in n-normed linear space $(X, \|.,\ldots,.\|)$. If $K = \{k_1 < k_2 < \ldots\} \in I$, then the subsequence $x_K = (x_{k_n})$ is called I-thin subsequence of the sequence x and if $K = \{k_1 < k_2 < \ldots\} \notin I$, then the subsequence $x_K = (x_{k_n})$ is called I-non- thin subsequence of the sequence x.

Theorem 3.4. Let $I \subset \mathcal{P}(X)$ be an admissible ideal in a n-normed linear space $(X, \|.,\ldots,.\|)$ and $x = (x_n)$ and $y = (y_n)$ are sequence in X such that $M = \{n \in \mathbb{N}: x_n \neq y_n\} \in I$. Then $I(\Lambda_x^n) = I(\Lambda_y^n)$ and $I(\Gamma_x^n) = I(\Gamma_y^n)$.
Proof. Let $M = \{n \in \mathbb{N}: x_n \neq y_n\} \in I$. If $\xi \in I(\Lambda_x^n)$. Then there is a set $K = \{m_1 < m_2 < \ldots\} \notin I$ such that $I - \lim_{k\to\infty} \| z_1, z_2, \ldots, z_{n-1}, x_{m_k} - \xi \| = 0$ for each non zero $z_1, z_2, \ldots, z_{n-1}$ in X.
Since $K_1 = \{n \in \mathbb{N}: n \in K, x_n \neq y_n\} \subset M \in I$, then $K_2 = \{n \in \mathbb{N}: n \in K, x_n = y_n\} \notin I$; because if $K_2 \in I$, then $K = K_1 \cup K_2 \in I$, but $K \notin I$. Hence the sequence $y_{K_2} = (y_{m_n})$ is a I-non- thin subsequence of $y = (y_n)$ and y_{K_2} converges to $\xi \in X$ i.e. $\xi \in I(\Lambda_y^n)$.Now if $\xi \in I(\Gamma_x^n)$, then $K_3 = \{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - \xi \| < \varepsilon\} \notin I$ for each $\varepsilon > 0$ and for each non zero $z_1, z_2, \ldots, z_{n-1}$ in X and $K_4 = \{n \in \mathbb{N}: n \in K_3, x_n = y_n\} \notin I$.Therefore $K_4 \subset \{n \in \mathbb{N}: \| z_1, z_2, ., z_{n-1}, y_n - \xi \| < \varepsilon\}$ for each $\varepsilon > 0$ and nonzero $z_1, z_2, \ldots, z_{n-1}$ in X. Thus it follows that for each $\varepsilon > 0$ and non zero $z_1, z_2, \ldots, z_{n-1}$ in X the set $\{n \in \mathbb{N}: |z_1, z_2, \ldots, z_{n-1}, y_n - \xi \| < \varepsilon\} \notin I$. i.e. $\xi \in I(\Gamma_y^n)$. This completes the proof.

The concepts of I and I^*-convergence are also introduced in different topological space (see [11],[13]). We can extend these concepts to the notion of the I^k-convergence for sequences in a n-normed linear space. In a analogue way we can introduce the definition of I and I^*-convergence for a sequences in a n-normed linear space.

Definition 3.5. Let $(X, \|.,\ldots,.\|)$be a n-normed linear space and I be an ideal on a set A. The function $f: A \to X$ is said to be I-convergent to $x \in X$ if for all nonzero $z_1, z_2, \ldots, z_{n-1}$in X and for all $\varepsilon > 0$ we have

$$A(\varepsilon) = \{a \in A: \| f(a) - x, z_1, z_2, \ldots, z_{n-1} \| \geq \varepsilon\} \in I.$$

We write it as $I - \lim f = x$.

Remark 3.6.If $A = \mathbb{N}$, we obtain the usual definition I-convergence of the sequence (x_n) to x in a n-normed linear spaced X.

Lemma 3.7. Let X, Y be two n-normed linear spaces and let A be a non empty set and I, $I_1$and I_2 be ideals on A. Then
(i) If I is not a proper ideal, then every function $f: A \to X$ is I-convergent to each point of X.
(ii) If $I_1 \subset I_2$, then for every function $f: A \to X$, we have $I_1 - \lim f = x \Rightarrow I_2 - \lim f = x$.

Proof. (i)Let x be a arbitrary element of X, then for all $\varepsilon > 0$ and for each non zero $z_1, z_2, \ldots, z_{n-1}$ in X, we have $A(\varepsilon) = \{a \in A: \| f(a) - x, z_1, z_2, \ldots, z_{n-1} \| \geq \varepsilon\} \in \mathcal{P}(A) = I$
(ii) Let $I_1 \subset I_2$ and $I_1 - \lim f = x$. Then we have for all $\varepsilon > 0$ and for all $z_1, z_2, \ldots, z_{n-1}$ in X the set
$A(\varepsilon) = \{a \in A: \| f(a) - x, z_1, z_2, \ldots, z_{n-1} \| \geq \varepsilon\} \in I_1 \subset I_2$. Hence $I_2 - \lim f = x$

Definition 3.8.$(X, \|.,\ldots,.\|)$ be a n-normed linear space and I be an ideal on $\mathbb{N}$. The sequence (x_n) in X is said to be $I^* -$ convergent to a point $x \in X$, if there exists a set $M \in \mathcal{F}(I)$such that $x_n \to x$ as $n \to \infty$ with respect to norm in M. We write it as $I^* - \lim x_n = x$. We now introduce the definition of $I^{\mathcal{K}}$-convergence. Now we replace the ideal Fin by an arbitrary ideal on the set A.

Definition 3.9. Let $(X, \|.,\ldots,.\|)$ be a n -normed linear space and let $\mathcal{K}$ and I be ideals on $\mathbb{N}$. The sequence (x_n) in X is said to be $I^{\mathcal{K}}$-convergence to a point $x \in X$, if there exists a set $M \in \mathcal{F}(I)$ and a sequence (y_n) such that

$$y_n = \begin{cases} x_n & \text{if } n \in M \\ x & \text{if } n \notin M \end{cases}$$ satisfying $\mathcal{K}$-$\lim y_n = x$. We write it as $I^{\mathcal{K}}$-$\lim x_n = x$.

Remark 3.10. The definition of $I^{\mathcal{K}}$-convergence can be reformulated in the form of decomposition. A sequence (x_n) is $I^{\mathcal{K}}$-convergence if and only if $(x_n) = (y_n) + (z_n)$ where (y_n) is $\mathcal{K}$-convergent and (z_n) is a sequence of non-zero elements on a set from I.

Example 3.11. We now exhibit some examples of ideals and their $I^{\mathcal{K}}$-convergence
(i) Let $I_0 = \mathcal{K}_0 = \{\phi\}$. I_0 is the minimal ideal in $\mathbb{N}$. A sequence (x_n) is $I^{\mathcal{K}}$-convergent if and only if it is constant
(ii) Let $\phi \neq M \subset \mathbb{N}, M \neq \mathbb{N}$. Take $\mathcal{K} = \mathcal{P}(M)$ i.e $\mathcal{K}$ is a proper ideal in $\mathbb{N}$. Let $I = \{\phi\}$. A sequence (x_n) is $I^{\mathcal{K}}$-convergent if and only if it is constant on $\mathbb{N}\backslash M$.
(iii) Let $\mathcal{K}$ be a admissible ideal in $\mathbb{N}$ and I be an arbitrary ideal. A sequence (x_n) is $I^{\mathcal{K}}$-convergent to a point $x \in X$ if there exists a set $M \in \mathcal{F}(I)$ and the sequence (y_n) given by Definition is such that the convergence of (y_n) is its usual convergence.

Theorem 3.12. The limit of any $I^{\mathcal{K}}$-convergent sequence in n-normed linear space X is unique.

Proof. Suppose that (x_n) be a $I^{\mathcal{K}}$-convergent sequence in a n-normed linear space X. Let $I^{\mathcal{K}} - \lim(x_n) = l_1$, $I^{\mathcal{K}} - \lim(x_n) = l_2$ and $l_1 \neq l_2$. Hence there exists $z_1, z_2, \ldots, z_{n-1}$ in X, such that $z_1, z_2, \ldots, z_{n-1}$ and $l_1 - l_2$ are linearly independent. Take $\varepsilon > 0$ such that

$$\| z_1, z_2, \ldots, z_{n-1}, l_1 - l_2 \| = 2\varepsilon.$$

Since $I^{\mathcal{K}} - \lim(x_n) = l_1$, by Definition 3.9 there exists a set $M_1 \in \mathcal{F}(I)$ such that the sequence (p_n) given by

$$p_n = \begin{cases} x_n & \text{if } n \in M_1 \\ l_1 & \text{if } n \notin M_1 \end{cases}$$

satisfies $\mathcal{K}$-$\lim p_n = l_1$.

Since $I^{\mathcal{K}} - \lim(x_n) = l_2$, therefore there exist $M_2 \in \mathcal{F}(I)$ such that the sequence (q_n) given by

$$q_n = \begin{cases} x_n & \text{if } n \in M_2 \\ l_2 & \text{if } n \notin M_2 \end{cases}$$

satisfies $\mathcal{K}$-$\lim q_n = l_2$.

Therefore for all $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1}$ in X. We have

$$\{m \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, p_m - l_1 \| \geq \varepsilon\} \in \mathcal{K}$$

$$\text{and } \{m \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, q_m - l_2 \| \geq \varepsilon\} \in \mathcal{K}.$$

Take $M = M_1 \cap M_2$. We have,

$$\begin{aligned} 2\varepsilon &= \| z_1, z_2, \ldots, z_{n-1}, l_1 - x_m + x_m - l_2 \| \\ &\leq \| z_1, z_2, \ldots, z_{n-1}, l_1 - x_m \| + \| z_1, z_2, \ldots, z_{n-1}, x_m - l_2 \| \\ &\leq \| z_1, z_2, \ldots, z_{n-1}, l_1 - p_m \| + \| z_1, z_2, \ldots, z_{n-1}, q_m - l_2 \| \end{aligned}$$

Therefore $\{m \in M: \| z_1, z_2, \ldots, z_{n-1}, q_m - l_2 \| < \varepsilon\} \subseteq \{m \in M: \| z_1, z_2, \ldots, z_{n-1}, p_m - l_2 \| \geq \varepsilon\} \in \mathcal{K}$, which is a contradiction to the fact that $I \neq \phi$.

We now show that $I^{\mathcal{K}}$-limit operation for the sequences in a n-normed linear space $(X, \|.,\ldots,.\|)$ is linear with respect to summation and scalar multiplication .

Theorem 3.13. Let (x_n) and (y_n) be sequences in n-normed linear space $(X, \|.,\ldots,.\|)$ and $I^{\mathcal{K}} - \lim(x_n) = l_1, I^{\mathcal{K}} - \lim(x_n) = l_2$, then (i) $I^{\mathcal{K}} - \lim(x_n + y_n) = l_1 + l_2$ (ii) $I^{\mathcal{K}} - \lim(\alpha x_n) = \alpha l_1$.

Proof. (i) Let $I^{\mathcal{K}} - \lim(x_n) = l_1$. By definition there exist $M_1 \in \mathcal{F}(I)$ such that the sequence (p_n) given by

$$p_n = \begin{cases} x_n & \text{if } n \in M_1 \\ l_1 & \text{if } n \notin M_1 \end{cases}$$

satisfies $\mathcal{K}$-$\lim p_n = l_1$.
Since $I^{\mathcal{K}} - \lim(x_n) = l_2$, there exist $M_2 \in \mathcal{F}(I)$ such that the sequence (q_n) given by

$$q_n = \begin{cases} x_n & \text{if } n \in M_2 \\ l_2 & \text{if } n \notin M_2 \end{cases}$$

satisfies $\mathcal{K} - \lim q_n = l_2$. Take $M = M_1 \cap M_2 \in \mathcal{F}(I)$ and we define a sequence

$$r_n = \begin{cases} x_n + y_n & \text{if } n \in M \\ l_1 + l_2 & \text{if } n \notin M. \end{cases}$$

and we see that $\mathcal{K} - \lim(p_n + q_n) = \mathcal{K} - \lim p_n + \mathcal{K} - \lim q_n = l_1 + l_2$
(ii) Proof is straightforward and left out.

Lemma 3.14. Let $\mathcal{K}$ and I be ideals on a set $\mathbb{N}$. Let (x_n) be sequences in a n-normed linear space $(X, \|.,\ldots,.\|)$ such that $\mathcal{K} - \lim(x_n) = l$. Then $I^{\mathcal{K}} - \lim(x_n) = l$.

Lemma 3.15. Let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2, I, I_1$ and I_2 be ideals in a set $\mathbb{N}$ such that $I_1 \subset I_2$ and $\mathcal{K}_1 \subset \mathcal{K}_2$. Let (x_n) be a sequences in a n-normed linear space $(X, \|.,\ldots,.\|)$ then we have

$$\text{(i) } I_1^{\mathcal{K}} - \lim x_n = l \Rightarrow I_2^{\mathcal{K}} - \lim x_n = l$$

$$\text{(ii) } I^{\mathcal{K}_1} - \lim x_n = l \Rightarrow I^{\mathcal{K}_2} - \lim x_n = l$$

Proof. (i) Suppose $I_1^{\mathcal{K}} - \lim x_n = l$,By definition there exist $M \in \mathcal{F}(I)$ such that the sequence (p_n) given by

$$p_n = \begin{cases} x_n & \text{if} n \in M \\ l & \text{if} n \notin M. \end{cases}$$

satisfies $\mathcal{K}$-$\lim p_n = l$.

Now for each $\varepsilon > 0$ and non zero $z_1, z_2, \ldots, z_{n-1}$ in X, we have

$$\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, p_n - l \| \geq \varepsilon\} \in \mathcal{K}$$

Since $I_1 \subset I_2$ we have $M \in \mathcal{F}(I_1) \subset \mathcal{F}(I_2)$. Therefore $I_2^{\mathcal{K}} - \lim x_n = l$

(ii) Suppose $I^{\mathcal{K}_1} - \lim x_n = l$, By definition there exist $M \in \mathcal{F}(I)$ such that the sequence (p_n) given by

$$p_n = \begin{cases} x_n & \text{if } n \in M \\ l & \text{if } n \notin M. \end{cases}$$

satisfies $\mathcal{K}$-$\lim p_n = l$.

For each $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1}$ in X. we have

$$\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, p_n - l \| \geq \varepsilon\} \in \mathcal{K}_1 \subset \mathcal{K}_2$$

Therefore $I^{\mathcal{K}_2} - \lim x_n = l$

In the followingt theorem, we show the relationship between the I-convergence and $I^{\mathcal{K}}$-convergence.

Theorem 3.16. Let $\mathcal{K}$ and I be ideals on a set $\mathbb{N}$.
Let (x_n) be sequences in n-normed linear space $(X, \|.,\ldots,.\|)$.
(i) If $I^{\mathcal{K}} - \lim(x_n) = l$ implies $I - \lim(x_n) = l$ for some $l \in X$, which has one neighborhood different from X, then $\mathcal{K} \subseteq I$
(ii) If $\mathcal{K} \subseteq I$ then $I^{\mathcal{K}} - \lim(x_n) = l$ implies $I - \lim(x_n) = l$

Proof. (i) Suppose that $\mathcal{K}$ is not a subset of I. Then there exists a set $V \in \mathcal{K}$ such that $V \notin I$. Let $l \in X$ has a neighborhood $U \subset X$ such that $U \neq X$ and $y \in X \backslash U$. We define a sequence (t_n) on X such that

$$t_n = \begin{cases} y & \text{if } n \in V \\ l & \text{if } n \notin V. \end{cases}$$

satisfying $\mathcal{K} - \lim(x_n) = l$. Thus by Lemma 3.14, we get $I^K - \lim(x_n) = l$.
Hence $\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, t_n - l \| \geq \varepsilon\} = V \notin I$. Hence the sequence (x_n) is not I- convergent to l

(ii) Let (x_n) be sequences in a n-normed linear space $(X, \|.,\ldots,.\|)$ and $l \in X$. Let $\mathcal{K} \subseteq I$ and $I^K - \lim(x_n) = l$. By definition of $I^{\mathcal{K}}$-convergence, there exist $M \in \mathcal{F}(I)$ such that the sequence (p_n) given by

$$p_n = \begin{cases} x_n & \text{if} n \in M \\ l & \text{if} n \notin M. \end{cases}$$

satisfying $\mathcal{K} - \lim(x_n) = l$. Now for all $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1}$ in X, we have

$$A(\varepsilon) = \{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, p_n - l \| \geq \varepsilon\} = \{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - l \| \geq \varepsilon\}$$

Hence $A(\varepsilon) \cap M \in \mathcal{K} \subseteq I$ and $\{n \in \mathbb{N}: \| z_1, z_2, \ldots, z_{n-1}, x_n - l \| \geq \varepsilon\} \subseteq (X \backslash M) \cup (A(\varepsilon) \cap M) \in I$

Therefore $I - \lim x_n = l$

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