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Application of Refuges and Harvesting on Prey-Prey-Predator System with Types I and II Functional Responses

Solomon Tolcha, Boka Kumsa and Purnachandra Rao Koya
Department of Mathematics, Wollega University, Nekemte, Ethiopia
Corresponding author: Purnachandra Rao Koya

***Abstract**: This paper studies about the dynamics of three population species interactions in biological ecology. Here, an interaction among two mutualistic preys and one predator populations has been considered. The population interaction areas are classified into two: free area and refuge area. In free area only the second prey and predator population species exist and interact while in a refuge area only the first prey population species exists. In the refuge area the predator population species cannot enter and attack the prey species. However, in the refuge area the two preys can interact and help each other. Additionally, in this model proportional harvesting function and functional responses are considered among these population interactions. Based on the unique and positive equilibrium points, local and global stability can be determined analytically and numerically. Simulation results supporting the analytical part are considered.*

Keywords: *Mutualism, Functional response, Local stability, Global stability, Harvesting function, Boundedness and Positivity.*

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I. Introduction

The Lotka -Volterra mathematical model describes mostly the Prey-predator system of population interaction in biological ecology [1-2]. In the population model, the well-known study of competing population interaction model is Lotka -Volterra two species competitive model. Based on the Lotka –Volterra mathematical model of population interaction different models of population interactions of two species have been considered and analyzed [3].

However, different scholars were considered and analyzed the mathematical model of three population species interactions in biological ecology theoretically and experimentally. The mathematical model of three population species interaction such as two predator and one prey population in which two predators competing for food on a single prey and another model which considered two prey and one predator population in which a single predator feeding on two competing prey species considered and analyzed. According to these studies, different criteria's were obtained for the system to be persists at the coexistence of equilibrium point (4).

To keep the persistence of the population species or the coexistence of species in ecology, there are different possible techniques such as stop harvesting, constructing reserved zones or refuges etc. which should be taken so that to save the species grow in the areas without disturbances. The roles of such kind of measures were studied by different scholars. However, the importance of reserved zones or refuges in prey-predator dynamics brings a major interest to the researchers. Here, the reserved zones or refuges can stabilize or unstabilize the coexistences of the population species in biological ecology [5–11].

A predator -prey model that can be incorporate a prey refuge and independent harvesting on either species was considered and studied by [17]. In this study, he observed and analyzed that harvesting function can destroy cyclic behavior of the dynamical system.

The role of reserved zone or a refuge on the dynamics of predator-prey system with Holling type I predator dependent functional response was investigated by the researcher called Dubey. According to this study, he concluded that the biologically interested equilibrium point or the positive equilibrium point, whenever exists, is always globally stable. This indicates that reserve zone has a stabilizing effect on the predator-prey population dynamics in ecology [6].

The work done by Dubey was modified by [12]. In this study, he considered and analyzed the dynamical behavior of a prey-predator population species with a reserved area where the predator functional response has been taken to be Holling type II functional responses.

In this paper, the interactions of two mutualistic prey populations and one predator have been considered. The populations are interacting in two areas: free area and refuge or a reserved area. According to this, it is considered that the second prey and the predator population exist and interact in free area while the first prey lives in a refuge area. In the free area, it is not possible for the predator to enter and attack the first prey.

Assumptions of the model

The following assumptions have been considered in order to construct the current model.

(a) There are three populations: two preys whose population densities are N_1 and N_2, and one predator whose population density is denoted by N_3.
(b) In absence of the predator the second prey population grows according to logistic law of growth.
(c) In absence of the second prey the predator population grows logistically.
(d) Two prey species help each other or there is mutualistic interaction between them.
(e) The first prey population helps the second prey population according to type I functional response and the second prey population helps the first prey population according to type II functional response.
(f) The first prey population lives in a refuge or in a reserved zone.
(g) It is impossible for the predator to enter and attack the first prey population.
(h) The second prey and the predator interact according to type II functional response.
(i) It is assumed that the predator population has alternative food. That is, the predator does not depend on the second prey alone for food to survive.
(j) The second population species are harvested with density dependent function or proportionally harvested.

The following flow diagram represents the interactions among the three species based on the above assumptions:

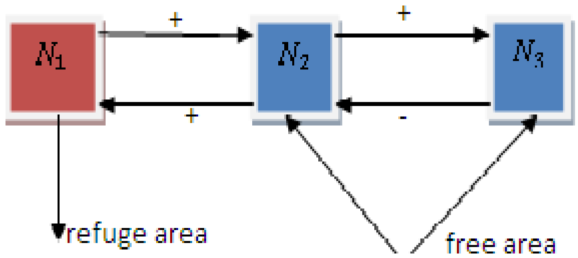

Figure 1 Interaction among two mutualistic preys N_1, N_2 and one predator N_3

The variables

The following variables are used in this model:

i. $N_1(\tau)$ – The density of the first population at time τ.
ii. $N_2(\tau)$ – The density of the second population at time τ.
iii. $N_3(\tau)$ – The density of the third population at time τ.

Here, the variables $N_1(\tau)$, $N_2(\tau)$ and $N_3(\tau)$ are dependent variables and time τ is independent variable. In this paper, for the simplicity we can let $N_1(\tau)$, $N_2(\tau)$ and $N_3(\tau)$ be represented as N_1, N_2 and N_3.

The parameters

The following parameters are used in this model:

i. The parameters r_1, r_2 and r_3 are the intrinsic growth rate of the first, second and third population respectively.
ii. The parameters a_{12} and a_{21} are the positive impacts of the second populations on the first population and vice versa respectively.
iii. h is helping time
iv. The parameters k_1, k_2 and k_3 are the carrying capacity of the first, second and third populations respectively.
v. The parameters q_2 and E_2 are the catch ability and the effort applying on the second population respectively.
vi. The parameters b_{23} and b_{32} are the negative impact of the third populations on the second population and positive impact of the second population on the third population respectively.

II. The model formulations

In this section, the three population species of two preys and a predator interaction have been described. This interaction consists of the interaction among two mutualistic preys and a predator. These populations interact in two areas: refuge and free area. From the model description, assumptions, definition of variables and parameters the dynamics of the three populations represent the following nonlinear differential equations:

$$\frac{dN_1}{d\tau} = r_1 N_1\left(1 - \frac{N_1}{k_1}\right) + \left[\frac{a_{12}N_1N_2}{1+a_{12}hN_1}\right] \tag{1}$$
$$\frac{dN_2}{d\tau} = r_2 N_2\left(1 - \frac{N_2}{k_2}\right) + a_{21}N_2N_1 - q_2E_2N_2 - \left[\frac{b_{23}N_3N_2}{1+b_{23}hN_2}\right] \tag{2}$$
$$\frac{dN_3}{d\tau} = r_3 N_3\left(1 - \frac{N_3}{k_3}\right) + \left[\frac{b_{32}N_3N_2}{1+b_{23}hN_2}\right] \tag{3}$$

The system of differential equations (1) – (3) represents the interaction of two mutualistic prey and predator population.

III. Normalization of the model

In order to reduce the number of parameters, we transform the system of equations (1) – (3) to the non dimensional form by using the following transformation of the variables. In this case let as assume $N_1 = k_1 x$, $N_2 = k_2 y$, $N_3 = k_3 z$ and $\tau = (1/r_1)\,t$ then, the original system of equation can be written as:

$$\frac{dx}{dt} = x\,(1-x) + \left(\frac{\alpha_{12}xy}{1+\delta_1 x}\right) \tag{4}$$
$$\frac{dy}{dt} = \delta_2 y\,(1-y) + \alpha_{21}yx - \delta_3 y - \left(\frac{\delta_{23}zy}{1+\delta y}\right) \tag{5}$$
$$\frac{dz}{dt} = \gamma_1 z\,(1-z) + \left(\frac{\delta_{32}zy}{1+\delta y}\right) \tag{6}$$

In the scaled equations (4) – (6) some notations are used to represent expressions as**:** $\alpha_{12} = (a_{12}k_2/r_1)$, $\delta_1 = (a_{12}hk_1)$, $\delta_2 = (r_2/r_1)$, $\alpha_{21} = (a_{21}k_1/r_1)$, $\delta_3 = (q_2E_2/r_1)$, $\delta_{23} = (b_{23}k_3/r_1)$, $\delta = b_{23}hk_2$, $\gamma_1 = (r_3/r_1)$, $\delta_{32} = (b_{32}k_2/r_1)$.

IV. Positivity of the solutions

Proposition 1 All solutions $x(t)$, $y(t)$ and $z(t)$ of the system of equation (4), (5) and (6) with positive initial conditions $(x_0, \ y_0, \ z_0)$ are positive for all $t \geq 0$.

Proof: The system of differential equation(4) can be given as, $dx/dt = x\left[1 - x + \left(\frac{\alpha_{12}y}{1+\delta_1 x}\right)\right]$. In this equation, x and y are the function of t. So by using change of variables and performing some algebraic manipulations, the solution of this differential equation can be obtained as: $x(t) = x_0\, exp\int_0^t\left[1 - \mathrm{x}(s) + \left(\frac{\alpha_{12}Y(s)}{1+\delta_1 X(s)}\right)\right]ds$. Here, x_0 is the density of the initial population of the first species at time $t = 0$. For every time t, the exponential function of $exp\int_0^t\left[1 - \mathrm{x}(s) + \left(\frac{\alpha_{12}Y(s)}{1+\delta_1 X(s)}\right)\right]ds$ is always positive. Therefore, the solution of $x(t) = x_0\, exp\int_0^t\left[1 - \mathrm{x}(s) + \left(\frac{\alpha_{12}Y(s)}{1+\delta_1 X(s)}\right)\right]ds$ of the model is positive for all t.

Similarly, the solution of an equation (5), $\frac{dy}{dt} = y\left[\delta_2 - \delta_2 y + \alpha_{21}x - \delta_3 - \left(\frac{\delta_{23}z}{1+\delta y}\right)\right]$ can be obtained as $y(t) = y_0\, exp\int_0^t\left[\delta_2 - \delta_2 y(s) + \alpha_{21}x(s) - \delta_3 - \left(\frac{\delta_{23}z(s)}{1+\delta y(s)}\right)\right]ds$. Here, y_0 is the initial population of the second species at $t = 0$. The exponential function in the solution of $y(t)$ is always positive for all time t. Therefore, the solution $y(t) = y_0\, exp\int_0^t\left[\delta_2 - \delta_2 y(s) + \alpha_{21}x(s) - \delta_3 - \left(\frac{\delta_{23}z(s)}{1+\delta y(s)}\right)\right]ds$ is always positive for all t.

Finally, consider the equation (6), $\frac{dz}{dt} = z\left[\gamma_1 - \gamma_1 z + \left(\frac{\delta_{32}y}{1+\delta y}\right)\right]$. The solution of this equation can be obtained as $z(t) = z_0\, exp\int_0^t\left[\gamma_1 - \gamma_1 z(s) + \left(\frac{\delta_{32}y(s)}{1+\delta y(s)}\right)\right]ds$. Here also the constant z_0 is the initial population of the third species at $t = 0$ and the exponential function is also positive for all t. Therefore, the solution $z(t) = z_0\, exp\int_0^t\left[\gamma_1 - \gamma_1 z(s) + \left(\frac{\delta_{32}y(s)}{1+\delta y(s)}\right)\right]ds$ is always positive for all t.

Hence, from the above three conditions the solutions $x(t)$, $y(t)$ and $z(t)$of the model equation (4) – (6) are positive for all $t \geq 0$.

V. Boundedness of the solutions

Lemma1. From the equation (4), the differential equation $dX/dt \leq x[1 - \mathrm{x}(1-\delta_1)]$ if $\delta_1 x^2 - \alpha_{12} y > 0$.

Proof: From the differential equation (4), the system of equation can be given as $\frac{dx}{dt} = x(1-x) + \left(\frac{\alpha_{12}xy}{1+\delta_1 x}\right)$. Now, this equation can be written as $\frac{dx}{dt} = \left[\frac{x[(1-x)(1+\delta_1 x)+\alpha_{12}y]}{1+\delta_1 x}\right]$. It is clear that $dx/dt \leq x[(1-x)(1+$ *δ1x+α12y= x1+δ1x−*x*1+δ1x+α12y*. Now, *dxdt=x[1−*x(1−*δ1*)*−(δ1x2−α12y)]≤x[1−*x(1−*δ1)]* if *δ1x2−α12y>0* is satisfied. Therefore, *dxdt≤x1−*x1−*δ1* if *δ1x2−α12y>0.*

Theorem 1: All solution of $x(t)$, $y(t)$ and $z(t)$ of the system of model equation (4) – (6) together with initial condition $(x_0,\ y_0,\ z_0)$ bounded with in a region $\Re = \{(x, y, z): 0 \leq x(t) \leq [1/(1-\delta_1)],\ 0 \leq y \leq$ *1+α21δ21−δ1, 0≤ zt≤1+ α21γ1δ+δ32γ1δ21−δ1* .

Proof: *Boundedness of* $x(t)$*:* From lemma 1, the differential equation $dx/dt \leq x[1 - \mathrm{x}(1-\delta_2)]$. So by using partial fraction method, the solution of $x(t)$ is given as: $x(t) \leq \{ce^t/[1 + (1-\delta_1)ce^t]\}$. Here, c is a constant obtained after substituting the initial condition and is given by $c = \{x_0/[1 - x_0(1-\delta_1)]\}$. Then, by using limit as $t \to \infty$ the solution $x(t)$ converges to $[1/(1-\delta_1)]$ for all $\delta_1 < 1$. This shows that the solution of the given equation is bounded for all $t \geq 0$.

Boundedness of $y(t)$*:* From the system of equation (5), $dy/dt = \delta_2 y(1-y) + \alpha_{21}yx - \delta_3 y - [\delta_{23}zy/(1+\delta y)]$. In fact this differential equation can be written as $dy/dt \leq y(\delta_2 + \alpha_{21}x - \delta_2 y)$. However, from the first model equation, we have the solution of $x(t) \leq [1/(1-\delta_1)]$. Thus, the solution of the equation (5) can be given as $y(t) \leq \left[\frac{ce^{\{\delta_2+[\alpha_{21}/(1-\delta_1)]\}t}\{\delta_2+[\alpha_{21}/(1-\delta_1)]\}}{1+c\delta_2 e^{\{\delta_2+[\alpha_{21}/(1-\delta_1)]\}t}}\right]$ where c is a constant obtain after substituting the initial condition and is given by $c = \left[\frac{y_0}{\{\delta_2+[\alpha_{21}/(1-\delta_1)]\}-y_0\delta_2}\right]$. Then, by using limit as $t \to \infty$ the solution $y(t)$ converges to $1 + \{\alpha_{21}/[\delta_2(1-\delta_1)]\}$ for all $\delta_1 < 1$. This shows that the solution is bounded from above for all $t \geq 0$.

Boundedness of $z(t)$*:* Similarly, from the equation (6), it is given that $dz/dt = \gamma_1 z(1-z) + [\delta_{32}zy/(1+\delta y)]$. In fact from the system of equation (6), $dz/dt \leq z(a - \gamma_1 z)$ where $a = \gamma_1 + (\gamma_1\delta + \delta_{32})y$ can be obtained. However, from the boundedness of the equation (5), $y = 1 + [\alpha_{21}/\delta_2(1-\delta_1)]$. The solution of the equation (6) can be given as $z(t) \leq [ace^{at}/(1 + ce^{at}\gamma_1)]$. Here, c is a constant obtain after substituting the initial condition and is given as $c = z_0/(a - z_0\gamma_1)$. Then, by using limit as $t \to \infty$ the solution $z(t)$ converges to $1 + [\alpha_{21}(\gamma_1\delta + \delta_{32})/\gamma_1\delta_2(1-\delta_1)]$ for all $\delta_1 < 1$.

Therefore, from the above all cases the solutions are bounded from above for all $t \geq 0$.

VI. The existence of equilibrium points

In this section, the conditions for the existence of the equilibrium points of the system can be identified and established as the following way.

1. $E_0 = (0,\ 0,\ 0)$ Trivial equilibrium point.
2. $E_1 = (1,\ 0,\ 0)$
3. $E_2 = (0,\ y_2,\ 0)$ where $y_2 = [1 - (\delta_3/\delta_2)] > 0$ if $\delta_2 > \delta_3$
4. $E_3 = (0,\ 0,\ 1)$
5. $E_4 = (x_4,\ y_4,\ 0)$. The predator population free equilibrium point Here, $x_4 = \{[\delta_3 + \delta_2(y_4 - 1)]/\alpha_{21}\} > 0\ if\ y_4 > 1\ or\ y_4 < 1\ and\ \delta_3 > \delta_2(y_4 - 1)$, $y_4 = (x_4 - 1)(1 + \delta_1 x_4)/\alpha_{12}$ for $x_4 > 1$ and $z_4 = 0$
6. $E_5 = (0,\ y_5,\ z_5)$ The first prey populations free equilibrium point. Here, $y_5 = [\gamma_1 z_5(z_5 - 1)/(\delta_{32} - \gamma_1\delta z_5 + \gamma_1\delta)] > 0$, $z_5 > 1$ and $\delta_{32} > \gamma_1\delta z_5 + \gamma_1\delta$ and $z_5 = (\delta_2 - \delta_2 y_5 - \delta_3)(1 + \delta y_5)/\delta_{23} > 0$, if $\delta_2 - \delta_2 y_5 - \delta_3 > 0$.
7. $E_6 = (1,\ 0,\ 1)$ The second prey population free equilibrium point
8. $E_7 = (x_7,\ y_7,\ z_7)$ Coexistence equilibrium point. Here, after some algebraic manipulation, the following solutions have been obtained: $y_7 = \{\gamma_1 z_7(z_7 - 1)/[\delta_{32} + \gamma_1\delta(1 - z_7)]\} > 0$ *where (* $\delta_{32} +$ *γ1δ >γ1δz7 , z7>1* and *x7=−b±b2−4ac2a* provided that *a=δ1, b=1−δ1* and *c=−(1+α12y7).* Here, there are two possibilities for x_7 : $x^*{}_1 = \{[-b + \sqrt{b^2 + 4ac}]/2a\} > 0$ for $b > 0$ or $x^*{}_2 = \{[b + \sqrt{b^2 + 4ac}]/2a\} > 0$ for $b < 0$. Since, $1 - \delta_1 > 0$ only the first solution can be the possible solution. Further, $z_7 = [(\delta_2 - \delta_2 y_7 - \delta_3 + \alpha_{21}x_7)(1 + \delta y_7)/\delta_{23}] > 0$ for $\delta_2 - \delta_2 y_7 - \delta_3 + \alpha_{21}x_7 > 0$.

VII. Local stability analysis

The local and asymptotically stability of an equilibrium point can be determined by constructing the community matrix for the model equation and finding the eigenvalues of the matrix at each equilibrium point. According to this, from the model equations (5) – (6), there is the community matrix that determines the dynamical behavior of the system and is given as following:

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix} \qquad (7)$$

Here in (7) the partial derivatives have the following expressions:

$\frac{\partial f}{\partial x} = 1 - 2x + \left[\frac{\alpha_{12} y}{(1+\delta_1 x)^2}\right]$; $\frac{\partial f}{\partial y} = \frac{\alpha_{12} x}{1+\delta_1 x}$; $\frac{\partial f}{\partial z} = 0$; $\frac{\partial g}{\partial x} = \alpha_{21} y$; $\frac{\partial g}{\partial y} = \delta_2 - 2\delta_2 \mathrm{y} + \alpha_{21} x - \delta_3 - \frac{\delta_{23} z}{(1+\delta y)^2}$; $\frac{\partial g}{\partial z} = \frac{\delta_{23} y}{1+\delta y}$; $\frac{\partial h}{\partial x} = 0$; $\frac{\partial h}{\partial y} = \frac{\delta_{32} z}{(1+\delta y)^2}$; $\frac{\partial h}{\partial z} = \gamma_1 - 2\gamma_1 z + \frac{\delta_{32} y}{1+\delta y}$.

Theorem 2: The trivial equilibrium point E_0 and the boundary equilibrium points E_1 , E_2 and E_3 are unstable.

Proof: *Stability of* E_0: The community matrix at the equilibrium point E_0 can be given as $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta_2 - \delta_3 & 0 \\ 0 & 0 & \gamma_1 \end{pmatrix}$. So, from the characteristics equation of this matrix, the following eigenvalues are obtained $\lambda_1 = 1 > 0$, $\lambda_2 = \delta_2 - \delta_3 > 0$ and $\lambda_3 = \gamma_1 > 0$. This indicates, all eigenvalues are positive which tell us that the system is unstable.

Stability of E_1: Similarly the community matrix at the equilibrium point E_1 is given as $J = \begin{pmatrix} -1 & \alpha_{12}/(1+\delta_1) & 0 \\ 0 & \delta_2 - \alpha_{21} - \delta_3 & 0 \\ 0 & 0 & \gamma_1 \end{pmatrix}$. From the characteristic equation of this matrix, an eigenvalues of $\lambda_1 = -1 < 0$, $\lambda_2 = \delta_2 - \alpha_{21} - \delta_3$ and $\lambda_3 = \gamma_1 > 0$ are obtained. Thus for every value of λ_2 the behavior of the system of equation at the given equilibrium point is saddle point which is unstable in general.

Stability of E_2: By following the same procedure, from the community matrix at the equilibrium point E_2 the following eigenvalues are obtained as $\lambda_1 = 1 + \alpha_{12}\, y_2 > 0$, $\lambda_2 = -\delta_2 + \delta_3 < 0$ and $\lambda_3 = \gamma_1 + [\delta_{32}\, y_2/(1 + \delta\, y_2)^2] > 0$. Therefore, from the values of an eigenvalues the system is stable in the direction of y and unstable in the direction of x and z. In general, the system at the given equilibrium point is saddle point which is unstable.

Stability of E_3*:* Similarly, from the community matrix at the equilibrium point E_3 there exists an eigenvalues of $\lambda_1 = 1 > 0$, $\lambda_2 = \delta_2 - \delta_3 - \delta_{23}$ and $\lambda_3 = -\gamma_1 < 0$. Thus, it is clear that the system is stable in the direction of z, unstable in the direction of x and unstable in the direction of y if $\delta_2 - \delta_3 - \delta_{23} > 0$ or stable in the direction of y if $\delta_2 - \delta_3 - \delta_{23} < 0$. Since, one eigenvalues is positive, the system of equation is saddle point which is unstable in general.

Theorem 3: The predators free equilibrium point E_4 and the first prey free equilibrium points E_5 are saddle point which are unstable in general.

Proof: *Stability of* E_4: The community matrix at the equilibrium point E_4 can be given as $J = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ where $a_1 = 1 - 2\, x_4 + [\alpha_{12}\, y_4/(1 + \delta_1\, x_4)^2] = x_4\, [\delta_1(1 - 2\, x_4) - 1] < 0$, $a_2 = [\alpha_{12}\, x_4/(1 + \delta_1\, x_4)] > 0$, $a_3 = 0 = a_7 = a_8$, $a_4 = \alpha_{21}\, y_4 > 0$, $a_5 = -\delta_2\, y_4 < 0$, $a_6 = [\delta_{23}\, y_4/(1 + \delta\, y_4)] > 0$, $a_9 = \gamma_1 + [\delta_{32}\ y_4/(1 + \delta\, y_4)] > 0$. Now, from the characteristics equation of this matrix, the following eigenvalues are obtained $\lambda_1 = \left[(a_1 + a_5) + \sqrt{(a_1 - a_5)^2 + 4a_2 a_4}\right]/2$, $\lambda_2 = \left[(a_1 + a_5) - \sqrt{(a_1 - a_5)^2 + 4a_2 a_4}\right]/2$ and $\lambda_3 = \gamma_1 + [\delta_{32}\, y_4/(1 + \delta\, y_4)] > 0$. Here, since both a_1 and a_5 are negative $a_1 + a_5 < 0$. So, the eigenvalues are $\lambda_1 > 0$ or, $\lambda_2 < 0$ and $\lambda_3 > 0$. To determine the stability of the system there are two options: (i) λ_1 , $\lambda_3 > 0$ and $\lambda_2 < 0$ (ii) λ_1 , $\lambda_2 < 0$ and $\lambda_3 > 0$. In both the cases the system is saddle point which is unstable in general.

Stability of E_5*:* Similarly, the community matrix at the equilibrium point E_5 can be given as $J = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ where $a_1 = 1 + \alpha_{12}\, y_5 > 0$, $a_2 = 0$, $a_3 = 0 = a_7$, $a_4 = \alpha_{21}\, y_5 > 0$, $a_5 = \{[-\delta_2\, y_5 + \delta\, y_5(\delta_2 - \delta_2\, y_5 - \delta_3)]/(1 + \delta\, y_5)\}$, $a_6 = [\delta_{23}\, y_5/(1 + \delta\, y_5)] > 0$, $a_8 = [\delta_{32}\, z_5/(1 + \delta\, y_5)^2] > 0$, $a_9 = \gamma_1 - 2\gamma_1\, z_5 + [\delta_{32}\, y_5/(1 + \delta\, y_5)]$. Thus, after performing some algebraic manipulations the eigenvalues are obtained as

$\lambda_1 = 1 + \alpha_{12}\, y_5 > 0$
$\lambda_2 = \left[(a_9 + a_5) + \sqrt{(a_5 - a_9)^2 + 4a_6 a_8}\right]/2$
$\lambda_3 = \left[(a_9 + a_5) - \sqrt{(a_5 - a_9)^2 + 4a_6 a_8}\right]/2$.
Here, there are three cases to determine the values of the eigenvalues of the above community matrix based on $a_9 + a_5$.

Case I. For $a_9 + a_5 > 0$, then, an eigenvalues of $\lambda_2 > 0$ and $\lambda_3 < 0\ or > 0$ are obtained. In this case there are two possibilities: λ_1 , λ_2 and $\lambda_3 > 0$ which is unstable or λ_1 , $\lambda_2 > 0$ and $\lambda_3 < 0$ which is saddle point.

Case II. For $a_9 + a_5 < 0$, the eigenvalues are $\lambda_2 > 0\ or < 0$ and $\lambda_3 < 0$. Here, again there are two possibilities based on the values of eigenvalues: $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$ which is a saddle point; and $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$ which is also a saddle point. However, in both the cases the dynamical system is saddle point which is unstable.

Case III. For $a_9 + a_5 = 0$ the eigenvalues are λ_1 , $\lambda_2 > 0$ and $\lambda_3 < 0$. So the system under this condition is also a saddle point.

Theorem 4: The second prey free equilibrium point E_6 is saddle point if $a_5 = \delta_2 - \delta_3 + \alpha_{21} - \delta_{23} > 0$ and locally asymptotically stable if $a_5 = \delta_2 - \delta_3 + \alpha_{21} - \delta_{23} < 0$.

Proof: *Stability of* E_6 :The community matrix at the given equilibrium point can be represented as $J = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ where $a_1 = -1 < 0$, $a_2 = [\alpha_{12}/(1 + \delta_1)] > 0$, $a_3 = a_6 = a_7 = a_4 = 0 =$ $a_5 = \delta_2 - \delta_3 + \alpha_{21} - \delta_{23} = a_8 = \delta_{32} > 0,\ a_9 = -\gamma_1 < 0$. Here, $\lambda_1 = -1 < 0$, $\lambda_{2,3} = \frac{(a_9 + a_5) \pm \sqrt{(a_5 - a_9)^2}}{2}$ which are $\lambda_2 = a_5 > or < 0$ and $\lambda_3 = a_9 < 0$. Here, based on the second eigenvalues λ_2 there are two cases:

i. $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$
ii. $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$

From the first case, it is clear that the system is saddle point and from the second case, since all eigenvalues are negative it is observed that the system is locally asymptotically stable.

Theorem 5: The coexistence equilibrium point E_7 is locally asymptotically stable.

Proof: *Stability of* E_7: The community matrix at the coexistence equilibrium point can be given as $J = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ where $a_1 = 1 - 2\, x_7 + \left[\frac{\alpha_{12}\, y_7}{(1+\delta_1\, x_7)^2}\right]$, $a_2 = \left(\frac{\alpha_{12}\, x_7}{1+\delta_1\, x_7}\right) > 0$, $a_3 = 0 = a_7$, $a_4 \alpha_{21}\, y_7 > 0$, $a_5 = \delta_2 - 2\delta_2\, y_7 + \alpha_{21}\, x_7 - \delta_3 - \left[\frac{\delta_{23}\, z_7}{(1+\delta\, y_7)^2}\right] = -\delta_2\, y_7 < 0$, $a_6 = \left[\frac{\delta_{23}\, y_7}{1+\delta\, y_7}\right] > 0$, $a_8 = \left[\frac{\delta_{32}\, z_7}{(1+\delta\, y_7)^2}\right] > 0$, $a_9 = \gamma_1 - 2\gamma_1\, z_7 + \left[\frac{\delta_{32}\, y_7}{1+\delta\, y_7}\right]$. In this case, the local stability of such kind of system can be obtained by using Routh – Hurwitz criteria. According to this criterion, the characteristics equation of the community matrix at the given equilibrium point can be written as:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \qquad (8)$$

Here in (8), $A = -(a_1 + a_5 + a_9)$, $B = -(a_6 a_8 - a_1 a_5 - a_1 a_9 - a_5 a_9 + a_2 a_4 + a_3 a_7)$ and $C = -(a_1 a_9 a_5 - a_1\, a_6 a_8 - a_2\, a_4 a_9 + a_2\, a_6 a_7 + a_3\, a_4 a_8 - a_3\, a_7 a_5)$. The objective of this criterion is, to find the roots of the characteristics equation without solving it. Here, the stability of the system can be determined by using a great mathematician Routh - Hurwitz criteria. According to this, if the root lies on the left half of a plane then the system is stable. In other word, from the equation (8), the system is stable if $A > 0$, $C > 0$ and $AB > C$.

VIII. Global Stability by Using Lyapunov Quadratic Function

Theorem 6: The coexistence equilibrium point E_7 is globally asymptotically stable for $-x^2 - z^2 + c < 0$

Proof: Consider the community matrix at the coexistence equilibrium point E_7 which can be given as:

$$X' = MX \qquad (9)$$

Here, in (9), the matrix notations used are $X' = (x'\ y'\ z')$, $M = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ and $X = (x,\ y,\ z)^T$. The objective here is to study the dynamical behavior of $X' = MX$ by Lyapunov direct method. Consider a quadratic Lyapunov function candidate as $V(X) = X^T pX$. Here p is a real symmetric positive definite matrix. Let the matrix p is of the form $p = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix}$. Now, Lyapunov equation of the form $pM + M^T p = -Q$ can be used to find the entries of p. Here Q is any symmetric positive matrix and without loss of generality it can be selected as $Q = I_3$. Thus, on solving Lyapunov equation and after performing some algebraic operations the entries of matrix p are obtained as:

$p_{11} = -(1/a_1)\,[(1/2) + p_{12}a_4]$, $p_{22} = -(1/a_5)[(1/2) + p_{13}a_2 + p_{33}a_8]$, $p_{33} = -(1/a_9)[(1/2) +$ *p23a6* , *p12=−1a1+a5p11a2+p13a8+p22a4* , *p13=−1a1+a9p12a6+p23a4* , *p23=−1a5+a9* $(p_{22}a_6+p_{33}a_8)$.

Here, it is straight forward to verify that the six elements p_{11} , p_{12} , p_{13} , p_{23} , p_{22} and p_{33} are all positive quantities. Recall that a matrix p is said to be a real symmetric positive definite matrix if the determinant of each of the minor of the matrix p are positive: That is, if

$|p_{11}| > 0\,, \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} > 0 \; for \; (p_{11}p_{22} - {p_{12}}^2) > 0$ and $\left|\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix}\right| > 0$ for $p_{11}(p_{33}p_{22} -$ *p232−p12p12p33−p13p23+p13p12p23−p13p22>0.*

Hence, the eigenvalues of matrix p also positive and thus the matrix p is a positive definite. So the Lyapunov function that is defined by the equation $V(X) = X^T pX$ and also can be expressed as:

$V(x\,y\,z) = x(xp_{11} + yp_{12} + p_{13}z) + y(xp_{12} + yp_{22} + p_{23}z) + z(p_{13}x + yp_{23} + zp_{33})$.

Here, it is observed that $V(x,\ y,\ z)$ is a positive function since $(x,\ \ y,\ z)$ is an interior equilibrium point $x,\ y$ and z are positive quantities.

Now, the time derivative of $V(X)$ is given by $V'(x,\ y,\ z) = (\partial V/\partial x)(dx/dt) + (\partial V/\partial y)(dy/dt) + (\partial V/\partial z)(dz/dt)$ and it reduces to the form as $V'(x,\ y,\ z) = -x^2 - z^2 + c$. Here $c = 2y^2(p_{12}a_2 + p_{23}a_8 +$ *p22a5+2yzp13a2.* Now, the Lyapunov function *V′x, y, z* is negative if

i. If $c < 0$ then the negativity of $V'(x,\ y,\ z)$ is trivial

ii. If $c > 0$ then $V'(x,\ y,\ z)$ is negative if $-x^2 - z^2 + c < 0$

Hence, the differential of Lyapunov function $V'(X)$ is negative if the above two conditions are satisfied.

Recall that an equilibrium point is said to be globally asymptotically stable if the Lyapunov function $V(X)$ satisfies the following three conditions on the entire state space:

(i) $V(X)$ is positive definite

(ii) its time derivative $V'(X)$ is negative and

(iii) The function $|V(X)| \to \infty$ as $||X|| \to \infty$.

Thus, the following result: It is all ready shown that the Lyapunov function $V(X)$ satisfies all the cited three conditions in case of the interior equilibrium point. Hence, E_7 is globally asymptotically stable based on the above two conditions.

IX. Numerical simulation

In this paper, we studied the dynamical behaviors of two mutualistic prey and one predator system with a prey refuge. Holling type I and type II functional response and proportional harvesting is taken to represent the interaction among the three population species. The coexistences of three population species are shown below by using numerical simulation at different initial population. Figure 2-4 shows the dynamical behavior of the three population species at different initial population with time t. Moreover, from the above simulation study we observed the following results

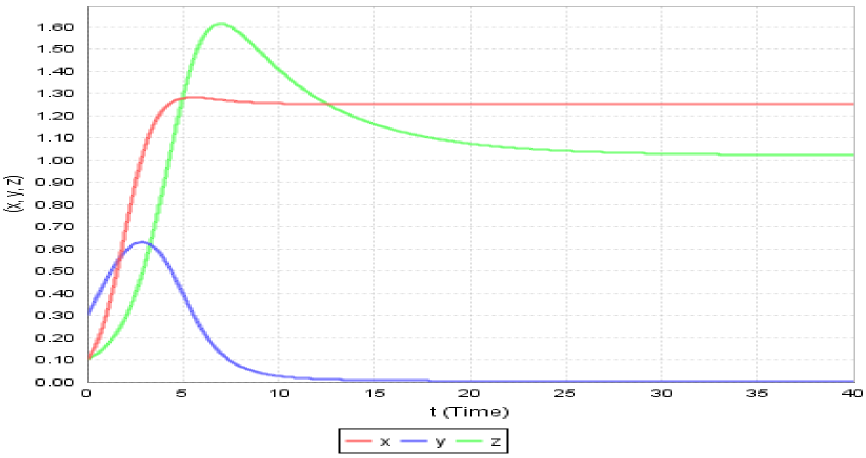

Figure 2 the dynamics of x, y and z versus time t at $\alpha_{12} = 0.1$, $\delta_1 = 0.25$, $\delta_2 = 1$, $\alpha_{21} = 0.1$, $\delta_3 = 0.2$, $\delta_{23} = 1$, $\delta = 0.5$, $\gamma_1 = 0.2$, $\delta_{32} = 1$

Figure 2 shows that the dynamics of three species at the initial population of $x = z = 0.1$ and $y = 0.3$. In this case, it is observed that starting from the initial population, the three population species grow in their densities but after some time t the second prey population y and the predator population z become equal.

Starting from this point, the second prey population decreases with high speed and become zero. However, the first population species x and the predator continue growing but after a time the first prey species reach its maximum growing while the predator again continue its growing up to its maximum values. After the z population reaches its maximum point, it starts to decline and become saturated. So that, the predator z population dominated by the first species population in which the dominance will continue forever.

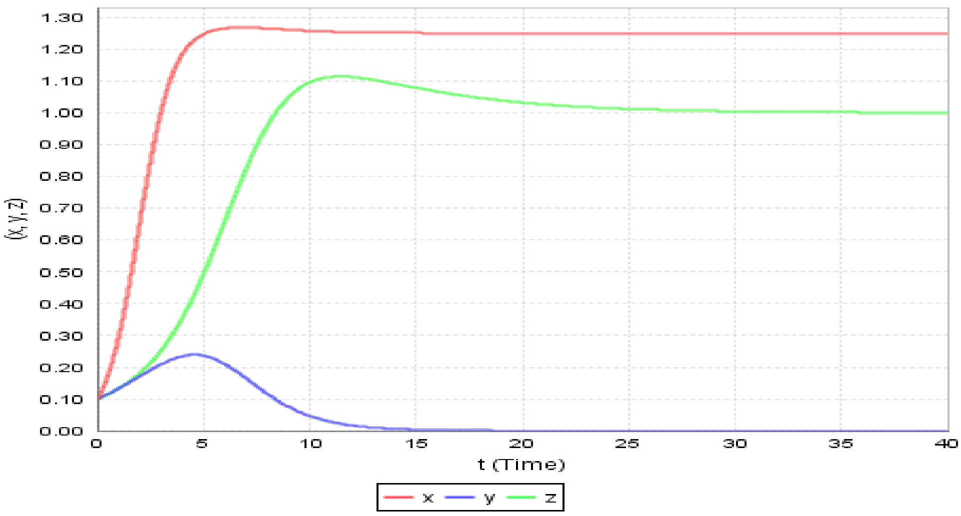

Figure 3 the dynamics of x, y and z versus time t at $\alpha_{12} = 0.1$, $\delta_1 = 0.25$, $\delta_2 = 1$, $\alpha_{21} = 0.1$, $\delta_3 = 0.5$, $\delta_{23} = 1$, $\delta = 0.5$, $\gamma_1 = 0.2$, $\delta_{32} = 1$.

Figure 3 shows that the dynamics of three population at the initial population of $x = y = z = 0.1$. Here, the initial populations of the three population species are equal. In this simulation the harvested values of the second species is increased. According to this, starting from the initial population all species attempt to grow but this will not go far because the size of the prey and the predator become equal and also the amount of the prey harvested is increased. This brings that the second prey species become zero. And also, since there is no interaction between the first prey species and the predator, both are growing up to their maximum point and become saturated.

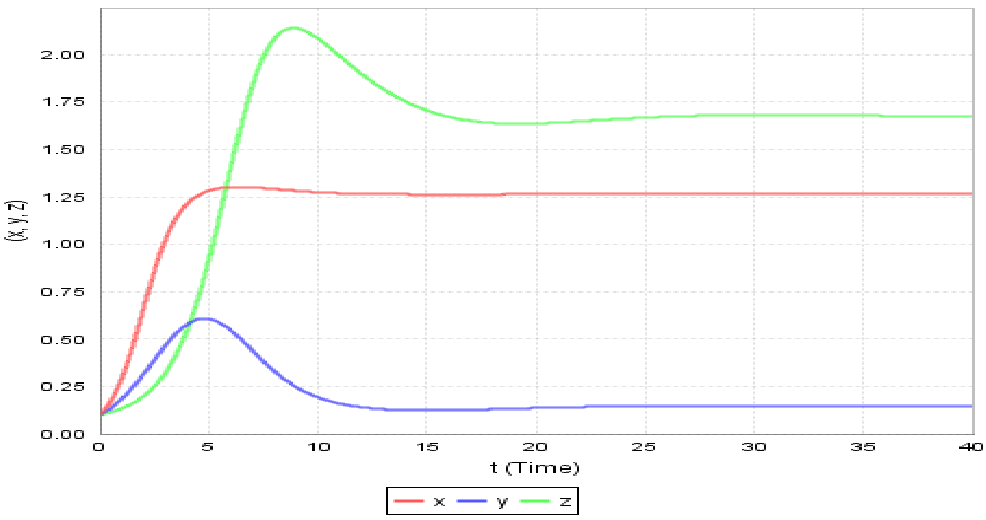

Figure 4 the dynamics of x, y and z versus time t when $\alpha_{12} = 0.1$, $\delta_1 = 0.25$, $\delta_2 = 1$, $\alpha_{21} = 0.1$, $\delta_3 = 0.2$, $\delta_{23} = 0.5$, $\delta = 0.5$, $\gamma_1 = 0.2$, $\delta_{32} = 1$

In figure 4, the initial populations of the three species are equal as figure 3. The difference is the negative impact of the z population on the y population is less when it compared with figure 2 and 3. So that from this simulation one can observes that, all population grow according to their intrinsic growth rate but within a few time the second prey y and the predator z becomes equal. However, starting from this point, the predator population continues its growing while the second prey population decline in its abundance. Here, the second prey species will not go to zero rather it become saturated at some constant value. The growth of the predator species continues until they reach a maximum population but after some time they start to declines and saturate.

X. Conclusion

In this paper, the ecological interactions of three population species were considered and analyzed. In these interactions, two mutualistic preys and one predator species were participated. Here, based on the interaction of these populations, the area of interactions divided into two patches: free area and a reserved area or a refuge. In the free area, only the y population and the z population were interacted while in a reserved area only the first prey species exist. But, it is impossible for the z population to enter and attack the x population. In this model, all population grows logistically.

According to this interaction, from the simulation result, the growth rate of the second prey population species y decreases and approaches to zero because this species is both harvested and attacked by the predator. If the negative impact of the z population over the y population is less than the positive impact of y population on z population, then, growth rate of the z population dominates the others forever otherwise the first x prey population dominate the z population. Since the first prey species is living in a refuge they grow up to their maximum point because the predator cannot control their growth in this area.

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