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Numerical Solutions For Singularly Perturbed Nonlinear Reaction Diffusion Boundary Value Problems

Hakkı DURU && Baransel GÜNEŞ
Corresponding Author: Hakkı DURU

Abstract: *In this article, the boundary value problem for singularly perturbed nonlinear reaction diffussion equations are treated. The exponentially fitted difference schemes on a uniform mesh which is accomplished by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder term in integral form are presented. The stability and convergence analysis of the method are discussed. The fully discrete scheme is shown to be convergent of order 1 in independent variable, independently of the perturbation parameter. Some numerical experiments have been carried out to validate the predicted theory.*

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I. Introduction

In this study, the problem of nonlinear reaction-diffusion boundary value problem is investigated:

$$-\varepsilon^2 u'' + a(x)u(x) = f(x,u) \tag{1}$$

$$u(0) = \kappa_0,\ u(l) = \kappa_1 \tag{2}$$

Such problems emerge as mathematical models of the research object in many areas of science. Reaction-diffusion equations are typical mathematical models in biology, physics and chemistry. Chemical reaction models are widely used in biological systems, population dynamics, and nuclear reactivity. These equations usually depend on various parameters such that temperature, catalyst, diffusion rate, etc. Moreover, they normally form a nonlinear dissipative system coupled by reaction between different substances [14].

Such equations also appear in the model of population distributions. J.G. Skellam's article on "Random dispersal in theoretical populations" has been a turning point [23]. A series of observations made by J.G. Skellam deeply affected his space ecology work. First, he set up a link between the theoretical biological species as a definition of movement, the random walk, and the diffusion equation as a definition of the distribution of the organism at the population density scale of species, and use the field data for the increase in the populations of the musk rats in Central Europe the connection is acceptable. Second, in parallel to Fisher's earlier contribution to genetics, he has presented reaction-diffusion equations in a theoretical ecologically effective manner by combining a common definition of distribution with population dynamics. Third, especially using both the various assumptions of linear (Malthusian) and logistic population growth rate terms, one and two dimensional habitat geometry, and interspace between the habitat and surrounding environmental regimes, Skellam studied reaction-diffusion models for a population distribution in a limited living space [5].

Reaction-diffusion mechanisms have been used to explain model formation in developmental biology and experimental chemical systems [19]. One of the ways to obtain oscillatory solutions in chemical system models is reaction-diffusion equations [3].

In the biological sense, morphogenesis, which is called chemical substances that diffuse into a tissue by entering the reaction, constitutes the main feature of morphogenesis. In this context, it is shown that the reaction-diffusion problems are utilized in the stability of the situations arising due to the morphogenesis suggested by Alan Turing [25].

Reaction and diffusion terms are used in synchronization methods due to the spatial binding of the particles in the regime of periodic or chaotic oscillations depending on the dominant force of dynamic behavior.

In recent times, reaction diffusion systems have received much attention as a prototype for model formation. Said models (facades, spirals, targets, hexagons, strips and scattering solitons) are used in a variety of reaction-diffusion systems.It has been argued that the reaction diffusion processes are a basis for processes dependent on morphogenesis in biology [10] and may even be related to animal pigmentation, skin pigmentation

[15], [18]. Other applications of reaction diffusion equations include ecological invasions [11], outbreak spread [17], tumor growth [6], [8], [22] and wound healing [21]. Another reason for the interest in reaction diffusion systems is the possibility of an analytical treatment, although there are nonlinear partial differential equations [7], [9], [12], [16], [24].

Singular perturbation problems include boundary layers that change rapidly in the solution function. In this type of problem, the classical difference schemes are not stable since the derivative of the solution is infinite at boundary layer. Also, the exact solution of such problems usually is not found. For this reason, numerical algorithms are needed. In this study, difference scheme with exponential coefficients are presented for singularly perturbed nonlinear reaction diffusion problems with boundary layer like [4]. In constructing these schemes, interpolation quadrature rules which are the remainder term in integral form and contain weight function are used [1]. The approach ratio of the solutions of the difference problems is $O(h^2)$.

II. Preliminary

In this section, some basic definitions and basic theorems as without proof will be given . The interpolating quadrature rules that are used in the construction of the difference scheme with their remainder term integral form and containing the base function are given. In addition, the quadrature formulas and notations needed to establish in a equidistance mesh are given. The required differential and integral inequalities and difference analogues will be given unspecified.

Lemma 1. Let $p(x)$ be a integrable function (weight function) and σ -a real parameter. Then, the following interpolating quadratic formulas are true:

$$\int_a^b f(x)p(x)dx = [\int_a^b p(x)dx]\,\{\sigma f(b) + (1-\sigma)f(a)\}$$

$$+f[a;b]\int_a^b\left(x - x^{(\sigma)}\right)p(x)dx + R(f) \tag{3}$$

$$R(f) = \int_a^b dxp(x)\int_a^b f^{(n)}(\xi)\,{K^*}_{n-1}(x,\xi)d\xi, n = 1\; or\; 2,$$

$$K_s(x,\xi) = T_s(x-\xi) - (b-a)^{-1}(x-a)(b-\xi)^s\,, \qquad s = 0,1,$$

$$x^{(\sigma)} = \sigma b + (1-\sigma)a\,,\; f[a,b] = \frac{f(b)-f(a)}{b-a},$$

$$T_s(\lambda) = \frac{\lambda^s}{s!}, \qquad \lambda \geq 0, \qquad T_s(\lambda) = 0, \qquad \lambda < 0.$$

In some cases, second term in (3) formula may add the remainder term:

$$\int_a^b f(x)p(x)dx = f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx + R^*(f) \tag{4}$$

$$R^*(f) = \int_a^b dxp(x)\int_a^b f^{(n)}(\xi)\,{K^*}_{n-1}(x,\xi)d\xi$$

$$+(n-1)f[a,b]\int_a^b (x - \frac{a+b}{2})p(x)dx, \qquad n = 1\; or\; 2,$$

$$K_n(x,\xi) = T_n(x-\xi) - T_n\left(\frac{a+b}{2} - \xi\right)$$

$$+(b-a)^{-1}(b-\xi)^S\left(\frac{a+b}{2} - x\right), \qquad s = 0,1.$$

Moreover,

$$\int_a^b f'(x)p(x)dx = f[a;b]\int_a^b p(x)dx + \bar{R}(f) \quad (5)$$

$$\bar{R}(f) = -\int_a^b dxp'(x)\int_a^b f^{(n)}(\xi)K_{n-1}(x,\xi)d\xi \,, n = 1,2.$$

In this formula, we determinate the same $K_s(x,\xi)$ function in (3) and (4) formulas, where

$$K_0(a,\xi) = K_0(b,\xi) = 0,$$

$$K_1(a,\xi) = K_1(b,\xi) = K_1(x,a) = K_1(x,b) = 0,$$

$$K_1(x,\xi) = K_1(\xi,x),$$

$$\frac{d}{dx}K_1(x,\xi) = K_0(\xi,x) \quad [1].$$

III. Asymptotic Estimations

For the (1)-(2) nonlinear problem, from the Taylor expansion, we can write

$$f(x,u) = f(x,0) + \frac{\partial f(x,\tilde{u})}{\partial u}u$$

where $\tilde{u} = \gamma u,\ 0 < \gamma < 1$, also we say $F(x) = f(x,0)$ and $b(x) = \frac{\partial f(x,\tilde{u})}{\partial u}$. Then

$$-\varepsilon^2 u'' + [a(x) - b(x)]u = F(x).$$

If we get

$$A(x) = a(x) - b(x)$$

we have

$$-\varepsilon^2 u'' + A(x)u = F(x) \quad (6)$$

where the following conditions valid:

$$A(x) = a(x) - \frac{\partial f(x,\tilde{u})}{\partial u} \geq \alpha > 0$$

$$F(x) = f(x,0) \geq 0.$$

We give these lemmas for the asymptotic estimates.

Lemma 2**.** Let $v(x)$ be a function and satisfy the following conditions:

$$Lv = F(x) \geq 0$$

$\kappa_0 \geq 0$, $\kappa_1 \geq 0$. Then $v(x) \geq 0$ [2].

Proof. To prove the lemma, assume that $v(x_0) = 0$, $v(x_1) = 0$ and $v(x_0) \leq 0$, for $x \in (x_0,x_1) \subset (0,l)$. Then

$$v''(\xi) = \frac{v(x_0) - 2v(\frac{x_0+x_1}{2}) + v(x_1)}{(\frac{x_1-x_0}{2})^2} \geq 0.$$

From this, it mean that $Lv < 0$ and $v(x) < 0$ for $x \in (x_0,x_1)$. This is contradictory to the hypothesis. So lemma is true.■

Lemma 3. For a $v(x) \in C[0,l] \cap C^2(0,l)$ function, the following estimation is true:

$$|v(x)| \le |v(0)| + |v(l)| + \alpha^{-1} \max_{0 \le s \le l} |Lv(s)|, \; 0 \le x \le l \tag{7}$$

[2].

Proof. For the proof of lemma, we consider the following barrier function

$$\psi(x) = \pm v(x) + |v(0)| + |v(l)| + \alpha^{-1} \max_{0 \le s \le l} |Lv(s)|, \qquad 0 \le x \le l.$$

For the boundary conditions, from this following relations

$$\psi(0) = \pm v(0) + |v(0)| + |v(l)| + \alpha^{-1} \max_{0 \le s \le l} |Lv(s)|, \qquad 0 \le x \le l$$

We get $\psi(0) \ge 0$ and from this following

$$\psi(l) = \pm v(l) + |v(0)| + |v(l)| + \alpha^{-1} \max_{0 \le s \le l} |Lv(s)|, \qquad 0 \le x \le l$$

$\psi(l) \ge 0$. Thus $L\psi(x) \ge 0$. From here $\psi(x) \ge 0$. From the relation (1), the inequality (7) is valid.■

Lemma 4. For the solution of the (1)-(2) nonlinear problem, the following estimations are true:

$$|u(x)| \le C, 0 < x < l \; ; A(x), F(x) \in C[0,l] \tag{8}$$

$$|u'(x)| \le C\left\{1 + \frac{1}{\varepsilon}\left(e^{\frac{-\sqrt{\alpha}x}{\varepsilon}} + e^{\frac{-\sqrt{\alpha}(l-x)}{\varepsilon}}\right)\right\}, 0 < x < l, \; A(x), \; F(x) \in C[0,l] \tag{9}$$

[2].

Proof. First, we show that $|u(x)| \le C$. From the relation (7)

$$|u(x)| \le |\kappa_0| + |\kappa_1| + \alpha^{-1} \max_{0 \le s \le l} |F(s)|,$$

then $|u(x)| \le C$. Now, we show that the estimation (9) is true. Taking into account the relation (8), if we modify the differential equation, we have the following estimation:

$$|u''(x)| = -\frac{1}{\varepsilon^2}|F(x) - A(x)u(x)| \le \frac{C}{\varepsilon^2}, \; 0 < x < l \tag{10}$$

Then, we needs to estimate for the values $|u'(0)|$ and $|u'(l)|$. Using the following formula:

$$g'(x) = g(\alpha_0;\alpha_1) - \int_{\alpha_0}^{\alpha_1} K_0(\xi,x) g''(\xi) d\xi, \qquad g \in C^2, \alpha_0 \le x \le l.$$

In this formula, if we get $g(x) = u(x)$, $x = 0$, $\alpha_0 = 0$, $\alpha_1 = \varepsilon$, then

$$|u'(0)| = \left|\frac{u(\varepsilon) - \kappa_0}{\varepsilon}\right| \le \frac{C}{\varepsilon} \tag{11}$$

In the same way, if we get $g(x) = u(x)$, $x = l$, $\alpha_0 = l - \varepsilon$, $\alpha_1 = l$, then

$$|u'(l)| = \left|\frac{\kappa_1 - u(l-\varepsilon)}{\varepsilon}\right| \le \frac{C}{\varepsilon} \tag{12}$$

Here, if we derivate the differential equation (6), then

$$-\varepsilon^2 u'' + A(x)u(x) = F(x)$$

$$-\varepsilon^2 u''' + A'(x)u(x) + A(x)u'(x) = F'(x).$$

we get

$$w(x) = u'(x)$$

and

$$\varphi(x) = F'(x) - [A'(x)u(x) + A(x)u'(x)],$$

then

$$Lw(x) = \Phi(x) \tag{13}$$

Moreover, from these relations (11) and (12),

$$w(0) = O\left(\frac{1}{\varepsilon}\right), w(l) = O\left(\frac{1}{\varepsilon}\right) \tag{14}$$

We search the solution of the linear problem (13)-(14) as the form $w(x) = w_0(x) + w_1(x)$, where the functions $w_0(x)$ and $w_1(x)$ respectively are the solutions of the following problems:

$$\begin{gathered} Lw_0 = \Phi(x), \qquad 0 < x < l \\ w_0(0) = w_0(l) = 0 \end{gathered} \tag{15}$$

and

$$\begin{gathered} Lw_1 = 0, \qquad 0 < x < l \\ w_1(0) = O\left(\frac{1}{\varepsilon}\right), \;\; w_1(l) = O\left(\frac{1}{\varepsilon}\right) \end{gathered} \tag{16}$$

For the solutions of the problem (15) according to the boundary conditions, we can write the following estimation:

$$|w_0(x)| \le \alpha^{-1} \max_{0 \le s \le l} |\Phi(s)|.$$

Therefore, if $\Phi(x)$ is uniformly bounded by ε, we have

$$|w_0(x)| \le C, \;\; 0 < x < l \tag{17}$$

Now, if we apply the maximum principle to the problem (16), then we get

$$|w_0(x)| \le \Lambda(x) \tag{18}$$

Here the function Λ(x) is the solution of the following problems:

$$\begin{gathered} -\varepsilon^2\Lambda'' + \alpha\Lambda = F(x), \qquad 0 < x < l \\ \Lambda(0) = |w_1(0)|, \;\; \Lambda(l) = |w_1(l)| \end{gathered} \tag{19}$$

The solution of the constant coefficient problem (19) is obvious:

$$\Lambda(x) = \frac{1}{\sinh(\frac{\sqrt{\alpha}l}{\varepsilon})}\left\{|w_1(0)|\sinh(\frac{\sqrt{\alpha}(l-x)}{\varepsilon} + |w_1(l)|\sinh(\frac{\sqrt{\alpha}x}{\varepsilon})\right\}.$$

From here, it easy true that the following estimation:

$$\Lambda(x) \le \frac{C}{\varepsilon}\left\{e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\varepsilon}}\right\} \tag{20}$$

This show that lemma is true. ■

IV. Construction Of Difference Schemes

4.1. Establishing the Exponential Fitted Difference Schemes. In this section, a difference scheme is established for the (1)-(2) non-linear reaction-diffusion problem is equidistant grid.

Let's make the following notations for this. Here, the following equidistant discrete point set say the equidistant grid:

$$\omega_h = \{x_i = ih, \qquad i = 1, \ldots, N-1\,, \qquad h = \frac{1}{N}\}, \qquad \varpi_h = \omega_h \cup \{0, l\}$$

The points x_i said the point of mesh. The functions defined on this grid are also called grid functions.$h = x_i - x_{i-1}$has called meshsize. For the grid function $u\colon \omega_h \to \mathbb{R}$ as $u(x_i) = u_i$, the following notations respectively are called the forward difference derivative, backward difference derivative and second difference derivative:

$$u_{x,i} = \frac{u_{i+1} - u_i}{h},$$

$$u_{\bar{x},i} = \frac{u_i - u_{i-1}}{h},$$

$$u_{\bar{x}x,i} = \frac{1}{h}\left(u_{x,i} - u_{\bar{x},i}\right)$$

[20]. We establish the difference scheme for the differential equation (1) on the uniform mesh ω_h. This scheme is the exponential fitted difference scheme. The difference scheme is established by using interpolation quadrature rules with the remaining terms in integral form and containing the weight function. For this purpose, we multiply both sides of the differential equation (1) with the base function φ_i, and integrate from x_{i-1} to x_{i+1}. Therefore, we have the following equation:

$$\chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu\varphi_i dx = \chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} \left(-\varepsilon^2 u'' + a(x)u(x)\right)\varphi_i dx$$

$$= \chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i dx,$$

where φ_i is the basis function and $\gamma_i = \frac{\sqrt{a_i}}{\varepsilon}$, $\varphi_i{}^{(1)}(x)$ and $\varphi_i{}^{(2)}(x)$ are respectively the solutions of the following problems:

$$\varepsilon^2 \varphi_i{}^{(1)''} - a_i\varphi_i{}^{(1)} = 0,\ \ \varphi_i{}^{(1)}(x_i) = 1, \varphi_i{}^{(1)}(x_{i-1}) = 0 \tag{21}$$

and

$$\varepsilon^2 \varphi_i{}^{(2)''} - a_i\varphi_i{}^{(2)} = 0,\ \ \varphi_i{}^{(2)}(x_i) = 1, \varphi_i{}^{(2)}(x_{i+1}) = 0 \tag{22}$$

The basis function $\varphi_i(x)$ is as follows

$$\varphi_i = \begin{cases} \varphi_i{}^{(1)}(x) = \dfrac{sinh\gamma_i(x - x_{i-1})}{sinh\gamma_i h}, & x \in (x_{i-1}, x_i) \\ \varphi_i{}^{(2)}(x) = \dfrac{sinh\gamma_i(x_{i+1} - x)}{sinh\gamma_i h}, & x \in (x_i, x_{i+1}) \end{cases}.$$

The function x_i is defined as follows

$$\chi_i = h^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i dx = \frac{2tanh\frac{\gamma_i h}{2}}{\gamma_i h}.$$

For the $-h^{-1}\int_{x_{i-1}}^{x_{i+1}} \varepsilon^2 u''\varphi_i dx$ term, if partial integration rule is used, it is found the following equation:

$$\chi_i{}^{-1}\left(h^{-1}\varepsilon^2 u'\varphi_i|_{x_{i-1}}^{x_{i+1}} - h^{-1}\varepsilon^2\int_{x_{i-1}}^{x_{i+1}} u'\varphi_i{}'dx\right)$$

$$= \chi_i{}^{-1}\left(h^{-1}\varepsilon^2\int_{x_{i-1}}^{x_i} u'\varphi_i{}^{(1)'}dx + \varepsilon^2\int_{x_i}^{x_{i+1}} u'\varphi_i{}^{(2)'}dx\right)$$

If interpolating quadrature rule (5) is applied to this result, it is obtained the following result:

$$\chi_i{}^{-1}h^{-1}\varepsilon^2\left\{u_{\bar{x},i}\int_{x_{i-1}}^{x_i}\varphi_i{}^{(1)'}dx + \int_{x_{i-1}}^{x_i}dx\varphi_i{}^{(1)''}\int_{x_{i-1}}^{x_i}\frac{d^2u(\xi)}{d\xi^2}T_1(x-\xi)d\xi\right.$$
$$\left. + u_{x,j}\int_{x_i}^{x_{i+1}}\varphi_i{}^{(2)'}dx + \int_{x_{i+1}}^{x_{i+1}}dx\varphi_i{}^{(2)''}\int_{x_{i+1}}^{x_{i+1}}\frac{d^2u(\xi)}{d\xi^2}T_1(x-\xi)d\xi\right\}$$
$$= -\chi_i{}^{-1}h^{-1}\varepsilon^2\left(u_{x,i} - u_{\bar{x},i}\right) + R_{1,i}(f) \qquad (23)$$

where $R_{1,i}(f)$ is defined as follows

$$R(f) = \chi_i{}^{-1}\left\{-\varepsilon^2\int_{x_{i-1}}^{x_i}dx\varphi_i{}^{(1)''}\int_{x_{i-1}}^{x_i}\frac{d^2u(\xi)}{d\xi^2}T_1(x-\xi)d\xi\right.$$
$$\left. - \varepsilon^2\int_{x_i}^{x_{i+1}}dx\varphi_i{}^{(2)''}\int_{x_i}^{x_{i+1}}\frac{d^2u(\xi)}{d\xi^2}T_1(x-\xi)d\xi\right\}.$$

For the $\chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} a(x)u(x)\varphi_i dx$ term, we obtain as follows:

$$\chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}a(x)u(x)\varphi_i dx = \chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}[a(x) - a(x_i) + a(x_i)]u\varphi_i dx$$

$$= \chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}a_i u\varphi_i dx + R^*{}_{a,i}$$

where

$$R^*{}_{a,i} = \chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}[a(x) - a(x_i)]u\varphi_i dx.$$

In addition to this result; if interpolation quadrature rule (4) is applied by taking as $\sigma = 1$ and $p(x) = \varphi_i{}^{(1)}$ in interval of $[x_{i-1}, x_i]$, $\sigma = 0$ and $p(x) = \varphi_i{}^{(2)}$ in interval of $[x_{i-1}, x_{i+1}]$, the following result is obtained:

$$\chi_i{}^{-1}h^{-1}[a_i u_i\int_{x_{i-1}}^{x_i}\varphi_i{}^{(1)}dx + u_x\int_{x_i}^{x_{i+1}}(x - x_i)\varphi_i{}^{(2)}dx$$

$$+ \int_{x_i}^{x_{i+1}}dx\varphi_i{}^{(2)}\int_{x_i}^{x_{i+1}}\frac{d^2u(\xi)}{d\xi^2}T_1(x-\xi)d\xi$$

$$+\int_{x_{i-1}}^{x_i} dx\varphi_i{}^{(1)} \int_{x_{i-1}}^{x_i} \frac{d^2u(\xi)}{d\xi^2} T_1(x-\xi)d\xi + u_i \int_{x_i}^{x_{i+1}} \varphi_i{}^{(2)}dx] + R^*{}_{a,i}$$

$$= -\chi_i{}^{-1}h^{-1}\{a_i u_i \int_{x_{i-1}}^{x_{i+1}} \varphi_i dx + a_i u_{\bar{x}} \int_{x_{i-1}}^{x_i} (x-x_i)\varphi_i{}^{(1)}dx$$

$$+a_i u_x \int_{x_i}^{x_{i+1}} (x-x_i)\varphi_i{}^{(2)}dx + a_i \int_{x_{i-1}}^{x_i} dx\varphi_i{}^{(1)} \int_{x_{i-1}}^{x_i} \frac{d^2u(\xi)}{d\xi^2} T_1(x-\xi)d\xi$$

$$+a_i \int_{x_i}^{x_{i+1}} dx\varphi_i{}^{(2)} \int_{x_i}^{x_{i+1}} \frac{d^2u(\xi)}{d\xi^2} T_1(x-\xi)d\xi\} + R^*{}_{a,i} \tag{24}$$

If we combine the expressions of (23) and (24), and consider the expressions of (1) and (2), we have the following expression

$$-\varepsilon^2\theta_i u_{\bar{x}x} + a_i u_i + R^*{}_{a,i} = \chi_i{}^{-1}h^{-1} \int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i dx \tag{25}$$

where

$$\theta_i = \chi_i{}^{-1}\left(1 + \varepsilon^{-2}a_i \int_{x_{i-1}}^{x_i} (x-x_i)\varphi_i{}^{(1)}dx\right) = \frac{(\gamma_i h)^2}{\left(sinh\frac{\gamma_i h}{2}\right)^2}.$$

For the $\chi_i{}^{-1}h^{-1} \int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i dx$ term, we obtain

$$\chi_i{}^{-1}h^{-1} \int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i dx = \chi_i{}^{-1}h^{-1} \int_{x_{i-1}}^{x_{i+1}} \{[f(x,u) - f(x_i,u)]\varphi_i dx$$

$$+\int_{x_{i-1}}^{x_{i+1}} [f(x_i,u) - f(x_i,u_i)]\varphi_i dx + f(x_i,u_i)\chi_i{}^{-1}h^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i dx\}$$

$$= f(x_i,u_i) + R_{f,i} \tag{26}$$

where the remainder term $R_{f,i}$ is as follows:

$$R_{f,i} = \chi_i{}^{-1}h^{-1}\left\{\int_{x_{i-1}}^{x_{i+1}} [f(x,u) - f(x_i,u)]\varphi_i dx + \int_{x_{i-1}}^{x_{i+1}} [f(x_i,u) - f(x_i,u_i)]\varphi_i dx\right\}.$$

If (21) and (22) are combined, following difference scheme is found:

$$\begin{gathered} -\varepsilon^2\theta_i u_{\bar{x}x,i} + a_i u_i = f(x_i,u_i) + R_i\,, \qquad i = 1,\dots,N-1 \\ u_0 = u_N = 0 \end{gathered} \tag{27}$$

where the remainder term is $R_i = R_{a,i} + R_{f,i}$.

For the approximate solution of y, we can write the following difference scheme

$$\begin{gathered} -\varepsilon^2\theta_i u_{\bar{x}x,i} + a_i u_i = f(x_i,u_i)\,, \qquad i = 1,\dots,N-1 \\ y_0 = y_N = 0 \end{gathered} \tag{28}$$

4.2. Error Estimations. If we get $z = y - u$, we can write the following problem:

$$-\varepsilon^2\theta_i z_{\bar{x}x,i} + A_i z_i = R_i\,, \qquad i = 1, \dots, N-1$$
$$z_0 = z_N = 0.$$

where if the mean value theorem is applied to this problem, following expression is found:

$$f(x_i, y_i) - f(x_i, u_i) = \frac{\partial f(x_i, \widetilde{u_i})}{\partial u}(y_i - u_i) \tag{29}$$

where

$$A_i = a_i - \frac{\partial f(x_i, \widetilde{u_i})}{\partial u} \tag{30}$$

If we consider (29) and (30), error of z satisfy the following boundary value difference problem:

$$lz_i = R_i, i = 1, N-1,\ z_0 = z_N = 0 \tag{31}$$

where the remainder term $R_i$is determined as $R_i = R_{a,i} + R_{f,i}$.

Theorem 5. For the $A(x) \in C^1[0, l]$, the solution of the difference problem (28) is uniformly convergence to the solution of the problem (1)-(2) according toε in ω_h. For the error, the following estimation is true:

$$\|y - u\|_{C(\omega_h)} \leq Ch$$

[2].

Proof. Here ε is arbitrary parameter and h is meshsize, from the maximum principle for the difference operator lv_i, if $lv_i \geq 0$, $(i = 1,2, \dots, N-1)$, $v_0 \geq 0$, $v_N \geq 0$, then it can be shown that $v_i \geq 0$, $i = 0,1, \dots, N$. If Lemma 3 is applied to the problem (31), we can write as follows

$$\|z\|_{C(\omega_h)} \leq \alpha^{-1}\|R\|_{C(\omega_h)} \tag{32}$$

Following estimation is obvious for R_i, from $R_{a,i}$ and $R_{f,i}$ on the $A(x) \in C^1[0, l]$,

$$|R_i| \leq Ch\ ,\ i = 1,2, \dots, N-1 \tag{33}$$

Proof of theorem is completed from (32) and (33).■

Remark 6. The difference problem (31) has only one solution. According to maximum principle, the following linear system

$$lv_i = 0, \qquad i = 1, N-1, \qquad y_0 = y_N = 0$$

has only null solution. As known, this is a necessary and sufficient condition for the existence of a unique solution for the system of linear equations [2].

Let's consider the necessary conditions for convergence ratio to be $O(h^2)$. We express this with a theorem.

Theorem 7. If $a(x) \in C^2[0, l]$, $f(x, u) \in C^2(D)$ and

$$a'(0) = a'(l) = 0 \tag{34}$$

the convergence ratio of the difference scheme (28) is $O(h^2)$ and we can write as follows:

$$\|y - u\|_{C(\omega_h)} \leq Ch^2 \tag{35}$$

[2].

Proof. We consider the remainder term $R_i = R_{a,i} + R_{f,i}$, for proof of theorem, we have

$$R_{a,i} = -\chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}[\,[a(x)-a(x_i)]u(x)\varphi_i(x)dx$$

$$R_{f,i} = \chi_i{}^{-1}h^{-1}\left\{\int_{x_{i-1}}^{x_{i+1}}[f(x,u)-f(x_i,u)]\varphi_i dx \;+\int_{x_{i-1}}^{x_{i+1}}[f(x_i,u)-f(x_i,u_i)]\varphi_i dx\right\}.$$

First, we show that

$$\left|R_{a,i}\right| = O(h^2),\; i=\overline{1,N-1} \tag{36}$$

If

$$a(x)-a(x_i) = (x-x_i)a'(x_i)+\frac{(x-x_i)^2}{2}a''(\xi_i), \qquad \xi_i\in(x_i,x)$$

and

$$u(x) = u(x_i)+(x-x_i)u'(\xi_i), \qquad \xi_i\in(x_i,x)$$

equalities are written in the term $R_{a,i}$, we obtain followings:

$$R_{a,i} = -\chi_i{}^{-1}h^{-1}\left\{a'(x_i)\int_{x_{i-1}}^{x_{i+1}}(x-x_i)u(x)\varphi_i dx-\frac{1}{2}\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2a''\big(\xi_i(x)\big)u(x)\varphi_i(x)dx\right\}$$

$$\begin{aligned}
&= -\chi_i{}^{-1}h^{-1}\left\{a'(x_i)u(x_i)\int_{x_{i-1}}^{x_{i+1}}(x-x_i)\varphi_i dx+a'(x_i)\int_{x_{i-1}}^{x_{i+1}}u'\big(\xi_i(x)\big)(x-x_i)^2\varphi_i(x)dx\right.\\
&\qquad\left.+\frac{1}{2}\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2a''\big(\xi_i(x)\big)u(x)\varphi_i(x)dx\right\}\\
&\qquad= \chi_i{}^{-1}h^{-1}a'(x_i)\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2u'\big(\xi_i(x)\big)\varphi_i(x)dx\\
&\qquad+\tfrac{1}{2}\chi_i{}^{-1}h^{-1}\textstyle\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2a''\big(\xi_i(x)\big)u(x)\varphi_i(x)dx
\end{aligned} \tag{37}$$

The second term on the right side of (37) satisfy the following inequality:

$$\left|\tfrac{1}{2}\chi_i{}^{-1}h^{-1}\textstyle\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2a''\big(\xi_i(x)\big)u(x)\varphi_i(x)dx\right| \le Ch^2, i=1,2,\dots,N-1 \tag{38}$$

from $\|a''(x)\|_{C[0,l]}\le C,\;\;\|u\|_{C[0,l]}\le C$ and $|x-x_i|^2\le h^2$. In order to evaluate the first term, if we replace following inequality

$$\begin{aligned}
|u'(\xi_i)| &\le C\left\{1+\frac{1}{\varepsilon}e^{\frac{-\sqrt{\alpha}\xi_i}{\varepsilon}}+\frac{1}{\varepsilon}e^{\frac{-\sqrt{\alpha}(l-\xi_i)}{\varepsilon}}\right\}\\
&\le C\left\{1+\frac{1}{\varepsilon}e^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}}+\frac{1}{\varepsilon}e^{\frac{-\sqrt{\alpha}(l-x_{i+1})}{\varepsilon}}\right\}, 1<i<N-1
\end{aligned}$$

with from the Lemma 4, the first term in (37), we obtain

$$\left|\chi_i{}^{-1}h^{-1}a'(x_i)\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2u'\big(\xi_i(x)\big)\varphi_i(x)dx\right|$$

$$\leq C\chi_i^{-1}h^{-1}|a'(x_i)|\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\varphi_i(x)dx$$

$$+\frac{1}{\varepsilon}C\chi_i^{-1}h^{-1}|a'(x_i)|\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\varphi_i(x)e^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}}dx$$

$$+\tfrac{1}{\varepsilon}C\chi_i^{-1}h^{-1}|a'(x_i)|\textstyle\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\varphi_i(x)e^{\frac{-\sqrt{\alpha}(l-x_{i+1})}{\varepsilon}}dx \qquad (39)$$

It can be easily seen that the convergence ratio of the first term on the right side of (39) is $O(h^2)$. The followings are seen from $a'(0)=0$ and $xe^{-x}<e^{-\frac{x}{2}}$, $(x\geq 0)$ to the second term:

$$\left|\frac{1}{\varepsilon}C\chi_i^{-1}h^{-1}|a'(x_i)|\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\varphi_i(x)e^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}}dx\right|$$

$$\leq\frac{1}{\varepsilon}C\chi_i^{-1}h^{-1}|a''(\xi_i)|x_ie^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}}\int_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\varphi_i(x)dx$$

$$\leq C_0h^2\frac{x_i}{\varepsilon}e^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}}\leq C_0h^2\frac{x_i}{x_{i-1}\sqrt{\alpha}}\frac{\sqrt{\alpha}x_{i-1}}{\varepsilon}e^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}}$$

$$\leq C_1h^2\frac{i}{i-1}e^{\frac{-\sqrt{\alpha}x_{i-1}}{2\varepsilon}}\leq Ch^2,\qquad i>1.$$

By the same way, the convergence ratio of the third term on the right side of (38) is $O(h^2)$ with $a'(l)=0$ (for $i<N-1$). So, the equality (37) is proved for $i=2,3,\dots,N-2$. For $i=1$ (by the same way for $i=N-1$), if we use

$$a(x)-a(x_1)=(x-x_1)a'(x_1)+\frac{(x-x_1)^2}{2}a''(\xi_1),\qquad \xi_1\in(x_1,x)$$

and

$$u(x)=u(x_0)+\int_{x_0}^{x}u'(\xi)\,d\xi,$$

the following is obtained:

$$R_{a,1}=-\chi_1^{-1}h^{-1}\{a'(x_1)\int_{x_0}^{x_2}(x-x_1)\left[\int_{x_0}^{x}u'(\xi)\,d\xi\right]\varphi_1dx$$

$$+\tfrac{1}{2}\chi_1^{-1}h^{-1}\textstyle\int_{x_0}^{x_2}(x-x_1)^2a''\big(\xi_1(x)\big)u(x)\varphi_1(x)dx\} \qquad (40)$$

The second term to the right of (40) is seen to be $O(h^2)$ from (38). We can evaulate the first term from $a'(0)=0$ and Lemma 4 as follows:

$$\chi_1^{-1}h^{-1}a'(x_1)\int_{x_0}^{x_2}(x-x_1)\left[\int_{x_0}^{x}u'(\xi)\,d\xi\right]\varphi_1dx$$

$$\leq|a'(x_1)|h\int_{x_0}^{x_2}|u'(x)|dx$$

$$\leq Cx_1 h|a''(\xi_1)| \int_{x_0}^{x_2} \left\{1 + \frac{1}{\varepsilon} e^{\frac{-\sqrt{\alpha} x_{i-1}}{\varepsilon}} + \frac{1}{\varepsilon} e^{\frac{-\sqrt{\alpha}(l - x_{i+1})}{\varepsilon}}\right\} dx$$

$$\leq C_1 h^2 \left\{h + \frac{1}{\varepsilon}\int_{x_0}^{x_2} e^{\frac{-\sqrt{\alpha}x}{\varepsilon}} dx\right\}$$

$$\leq C_1 h^2 \left\{h + (\alpha)^{-1}\left(1 - e^{\frac{-2\sqrt{\alpha}h}{\varepsilon}}\right)\right\} = O(h^2)$$

Thus we prove $\left|R^{(1)}{}_{a,i}\right| = O(h^2)$, $\left|R^{(N-1)}{}_{a,i}\right| = O(h^2)$. So (40) satisfied.

Now we consider term $R_{f,i}$, $u_m = \underset{[0,l]}{min}|u(x)|$, $u_M = \underset{[0,l]}{max}|u(x)|$, $(x,u) \in D = [0,l] \times [u_m, u_M]$, $f(x,u) \in C^2(D)$, it follows

$$R_{f,i} = \chi_i{}^{-1}h^{-1}\{\int_{x_{i-1}}^{x_{i+1}}[f(x,u(x)) - f(x_i,u(x))]\varphi_i(x)dx$$

$$+\int_{x_{i-1}}^{x_{i+1}}[f(x_i,u(x)) - f(x_i,u(x_i))]\varphi_i(x)dx\}$$

Using closed derivative formula and Taylor expansion, we have

$$f(x,u(x)) - f(x_i,u(x)) = (x - x_i)\left[\frac{\partial f(x_i,u_i)}{\partial x} + \frac{\partial f(x_i,u_i)}{\partial u}\frac{\partial u(x_i)}{\partial x}\right]$$

$$+\frac{(x-x_i)^2}{2!}\left[\frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial x^2} + 2\frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial x\partial u}\frac{du(\xi_i)}{dx} + \frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial u^2}\left(\frac{du(\xi_i)}{dx}\right)^2\right.$$

$$\left. + \frac{\partial f(\xi_i,u(\xi_i))}{\partial u^2}\frac{d^2u(\xi_i)}{dx^2}\right].$$

From here

$$R_{f,i} = \chi_i{}^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}(x - x_i)\left[\frac{\partial f(x_i,u_i)}{\partial x} + \frac{\partial f(x_i,u_i)}{\partial u}\frac{\partial u(x_i)}{\partial x}\right]\varphi_i(x)dx$$

$$+\int_{x_{i-1}}^{x_{i+1}}\frac{(x-x_i)^2}{2!}\left\{\frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial x^2} + 2\frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial x\partial u}\frac{du(\xi_i)}{dx}\right.$$

$$\left. + \frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial u^2}\left(\frac{du(\xi_i)}{dx}\right)^2 + \frac{\partial f(\xi_i,u(\xi_i))}{\partial u^2}\frac{d^2u(\xi_i)}{dx^2}\right\}\varphi_i(x)dx,$$

$$\left|R_{f,i}\right| = O(h^2) \tag{41}$$

is obvious from Lemma 4, first and second derivative estimation for u function. Finally, it is seen that

$$\|R_i\|_{C(\omega_h)} \leq Ch^2$$

from (36) and (41) in the following equality

$$R_i = R_{a,i} + R_{f,i},$$

Theorem is proved from this and (32) [2].

V. Numerical Example

The difference schemes in this section tested on the following non-linear problem:

$$-\varepsilon^2 u''(x) + x(1-x)u(x) = u + u^2, \qquad x \in (0,1)$$
$$u(0) = u(1) = 0 \tag{42}$$

Let's write this problem explicitly for the (27) and (28):

$$-\varepsilon^2 h^{-2}\theta_i\left(y_{i+1}{}^{(n)} - 2y_i{}^{(n)} + y_{i-1}{}^{(n)}\right) + a_i y_i{}^{(n)} - y_i{}^{(n)}\frac{\partial f(x_i, y_i{}^{(n-1)})}{\partial u}$$
$$= f\left(x_i, y_i{}^{(n-1)}\right) - y_i{}^{(n-1)}\frac{\partial f\left(x_i, y_i{}^{(n-1)}\right)}{\partial u}.$$

We edit this according to following:

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i\,, i = 1, \dots, N-1$$
$$y_0 = y_N = 0$$

Here the initial iteration is$y_i{}^{(0)} = -2, i = 1, \dots, N-1$. Also

$$A_i = B_i = \varepsilon^2 h^{-2}\theta_i,$$

$$C_i = 2\varepsilon^2 h^{-2}\theta_i + a_i - \frac{\partial f\left(x_i, y_i{}^{(n-1)}\right)}{\partial u},$$

$$F_i = f\left(x_i, y_i{}^{(n-1)}\right) - y_i{}^{(n-1)}\frac{\partial f\left(x_i, y_i{}^{(n-1)}\right)}{\partial u}.$$

The elimination method and iteration should be applied together for the sample. $C_i - \alpha_i A_i \neq 0$ and the elimination method is defined by

$$\alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i}\,, \alpha_1 = 0, i = 1, \dots, N-1$$

$$\beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - \alpha_i A_i}\,, \beta_1 = 0, i = 1, \dots, N-1$$

and $y_i = y_{i+1}\alpha_{i+1} + \beta_{i+1}, i = N-1, \dots, 0$ [20].

Absolute errors are determined as

$$r_0 = \max_{0 < i < N}\left|y^i{}_h - y^i{}_{\frac{h}{2}}\right|, r_1 = \max_{0 < i < N}\left|y^i{}_{\frac{h}{2}} - y^i{}_{\frac{h}{4}}\right|$$

because the analytical solution of the test problem is not known. The smooth convergence ratio is calculated as follows:

$$p = \frac{ln\frac{r_1}{r_2}}{ln2}$$

[13]. Results are presented in following tables.Results in the solution of (42) test problem are calculated on uniform mesh. Errors and p convergence ratios are given on Table (1)-(2).

Table 1. Convergence ratio at a uniform mesh points for $\varepsilon = 2^{-w}$ $(w = 3, 4, \ldots, 9)$

ε	N=8	N=16	N=32	N=64
2^{-3}	r0=0.311360	r0=0.046374	r0=0.009127	r0=0.002110
	r1=0.042791	r1=0.009127	r1=0.002110	r1=0.002110
	p=2.863215	p=2.345066	p=2.112825	p= 2.030424
2^{-4}	r0=0.228723	r0=0.302153	r0=0.045168	r0=0.009008
	r1= 0.181690	r1=0.042560	r1=0.009008	r1=0.002093
	p=0.332122	p= 2.827703	p=2.326072	p=2.105377
2^{-5}	r0=0.029511	r0=0.213193	r0=0.293582	r0=0.044029
	r1=0.021707	r1=0.173294	r1=0.042189	r1=0.008846
	p=0.443117	p=0.298938	p=2.798826	p=2.315451
2^{-6}	r0=0.013264	r0=0.034541	r0=0.202879	r0=0.287519
	r1=0.003572	r1=0.019164	r1=0.167515	r1=0.041926
	p=1.892868	p=0.849931	p=0.276334	p=2.777732
2^{-7}	r0=0.001015	r0=0.017297	r0=0.037469	r0=0.197143
	r1=0.000229	r1=0.003867	r1=0.017750	r1=0.164258
	p=2.145248	p=2.161150	p=1.077877	p=0.263277
2^{-8}	r0=0.000009	r0=0.002291	r0=0.020158	r0=0.039032
	r1=0.000015	r1=0.000280	r1=0.004050	r1=0.017014
	p=-0.780445	p=3.032668	p=2.315307	p=1.197961
2^{-9}	r0=0.000000	r0=0.000048	r0=0.003855	r0=0.021867
	r1=0.000001	r1=0.000008	r1=0.000335	r1=0.004149
	p=-5.619925	p=2.529604	p=3.526566	p=2.397955

Table 2. Convergence ratio at a uniform mesh points for $\varepsilon = 2^{-w}$ $(w = 3, 4, \ldots, 9)$

ε	N=128	N=256	N=512	N=1024
2^{-3}	r0=0.000518	r0=0.000129	r0=0.000032	r0=0.000008
	r1=0.000129	r1=0.000032	r1=0.000008	r1=0.000002
	p=2.008042	p=2.002020	p=2.000501	p= 2.000123
2^{-4}	r0=0.002103	r0=0.000516	r0=0.000128	r0=0.000032
	r1= 0.000515	r1=0.000128	r1=0.000032	r1=0.000008
	p=2.030544	p= 2.007474	p=2.001877	p=2.105377
2^{-5}	r0=0.008854	r0=0.002081	r0=0.000510	r0=0.00012
	r1=0.002061	r1=0.000510	r1=0.000127	r1=0.000032
	p=2.102758	p=2.029223	p=2.007438	p=2.001832
2^{-6}	r0=0.043257	r0=0.008806	r0=0.002068	r0=0.000507
	r1=0.008739	r1=0.002041	r1=0.000507	r1=0.000126
	p=2.307372	p=2.109253	p=2.0282252	p=2.007188
2^{-7}	r0=0.284069	r0=0.042822	r0=0.008779	r0=0.002060
	r1=0.041776	r1=0.008680	r1=0.002030	r1=0.000505
	p=2.76549	p=2.302650	p=2.112542	p=2.027695
2^{-8}	r0=0.194140	r0=0.282245	r0=0.042593	r0=0.008765
	r1=0.162543	r1=0.041696	r1=0.008648	r1=0.002028
	p=0.256272	p=2.758954	p=2.300122	p=2.111332
2^{-9}	r0=0.039837	r0=0.192606	r0=0.281309	r0=0.42476
	r1=0.016639	r1=0.161665	r1=0.041655	r1=0.008632
	p=1.259558	p=0.252646	p=2.755580	p=2.298817

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