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A Concept of Double $\alpha\beta$-Statistical Convergence of Order γ in Probability

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ABSTRACT: *In this paper, we shall introduce the extension of $\alpha\beta -$ statistical convergence of order γ in probability to double sequences. The study will also establish some inclusion relations between $\alpha\beta -$ statistical convergences of order γ and $\alpha\beta -$ statistical convergences of order γin probabilityof double sequences. Finally, we will give a condition under which a double sequence of random variables will converge to a unique limit under two different (α, β) of double sequences and also to prove that if this condition violates then the limit value of $\alpha\beta -$ statistical convergence of order γ in probability of double sequence of random variables for two different (α, β) of double sequence may not be equal.*
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I. Introduction

The idea of convergence of a real sequence has been extended to statistical convergence by Fast (1951) and Steinhaus (1951) and later on reintroduced by Shoenberg (1959) independently and is based on the notion of asymptotic density of the subset of natural numbers. However, the first idea of statistical convergence (by different name) was given by zygmund (1979) in the first edition of his monograph published in Warsaw in 1935. Later on it was further investigated from the sequence space point of view and linked with summability theorem by Fridy (1985), Das *et al.*(2011). Friday and Orhan (1993).

In Bhunia, Das and Pal (2012) and Colak (2010), a different direction was given to the study of statistical convergence of order $\gamma(0 < \gamma < 1)$ was introduced by using the notion of natural density γ (where n is replaced by n^{γ} in the denominator in the definition of natural density). It was observed in Bhunia, Das and Pal (2012), that the behavior of this new convergence was not exactly parallel to that of statistical convergence and some basic properties were obtained. More results on this convergence can be seen from Seng$\ddot{u}$l and Et (2014).

Quiet recently, Das *et al*. (2018) introduced and studied the concept of $\alpha\beta -$ statistical convergence of order γ in probability for single sequence. Following their introduction, we shall further extended this concept to double sequence and establish some inclusion relation between $\alpha\beta -$ statistical convergence of order γ and $\alpha\beta -$ statistical convergence of order γ in probability.

Definition 1.1 (Fast and Steinhaus [1951]): A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to L if for arbitrary,$\varepsilon > 0$ the set

$$K(\varepsilon) = \{ n \in \mathbb{N} : |x_n - L| \geq \varepsilon \} = 0$$

Has a natural density zero.

Definition 1.2 (Mursaleen and Edely [2003]): A real double sequence $\{x_{jk}\}_{j,k\in N}$ is statistically convergent to a number L if for each,$\varepsilon > 0$, the set $\{ (j,k), j \leq m \text{ and } k \leq n: \left|x_{jk} - L\right| \geq \varepsilon \}$ has double natural density zero. In this case, we write $\mathrm{S}t_2 - \lim_{j,k} x_{j,k} = L$.

Definition 1.3 (Colak [2012]): A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent of order γ where $(0 < \gamma \leq 1)$ to a real number x if for each,$\varepsilon > 0$, the set

$K = \{ n \in \mathbb{N} : |x_n - L| \geq \varepsilon \}$ has γ– natural density of zero i.e

$\lim_{n\to\infty} \frac{1}{n^{\gamma}} |\{K \leq n: |x_k - x| \geq \varepsilon\}| = 0$ and we write $x_n \overset{S^{\gamma}}{\rightarrow} x$.

Definition 1.4 (Aktuglu [2014]): A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\alpha\beta -$ statistical convergence of order γ where $(0 < \gamma \leq 1)$ to a real number x if for each $\varepsilon > 0$, the set $\{ n \in \mathbb{N}: |x_k - x| \geq \varepsilon\}$ has $\alpha\beta -$ natural density zero i.e

$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)^{\gamma}} | K \in [\alpha_n, \beta_n]: |x_k - x| \geq \varepsilon\}|$ and we write $S_{\alpha\beta}^{\gamma} - lim\, x_n = 0$ or $x_n \xrightarrow{S_{\alpha\beta}^{\gamma}}$ x.

II. $\alpha\beta-$ Statistical Convergence Of Order γ In Probability

The following definitions and results are by Das *et al.* (2018).

Definition 2.1(Das *et al.*, [2018]): Let (S,Δ,P) be a probability space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to given class of events Δ and a given probability function P: $\Delta\rightarrow\mathbb{R}$. Then the sequence$\{x_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta-$ statistical convergence of order γ(where $0<\gamma\leq 1$) in probability to a random variable$X$ ($where\ X: S\rightarrow\mathbb{R}$) if for any $\varepsilon,\delta>0$

$\lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)^{\gamma}}\,|\,\{k\in[\alpha_n,\beta_n]: P(|X_k-X|)\geq\ \varepsilon)\geq\delta\}|=0$

Or equivalently,

$\lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)^{\gamma}}\,|\,\{k\in[\alpha_n,\beta_n]: 1-P(|X_k-X|<\ \varepsilon)\geq\delta\}|=0.$

Definition 2.2. (Das *et al.*, [2018]):Let (S,Δ,P) be a probability space and $\{X_n\}_{n\in\mathbb{N}}$be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events Δ and a given probability function$P:\Delta\rightarrow\mathbb{R}$. A sequence of random variables$\{X\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$ $-$strong p-Cesàro summable of order γ where ($0<\gamma\leq 1$) and p > 0 is any fixed positive real number) in probability to a random variable X if for any $\varepsilon>0$

$\lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)^{\gamma}}\sum_{k\in[\alpha_n,\beta_n]}\{P(|X_k-X|\geq\ \varepsilon)^p\}=0.$

In this case we write $X_n\xrightarrow{\mathrm{p}W_{\alpha\beta}^{\gamma,p}}$ X. The class of all sequences of random variables which are $\alpha\beta$-strong p-Cesàro summable of order γ in probability is denoted simplyby p$W_{\alpha\beta}^{\gamma,p}$.

Definition 2.3 (Das *et al.*, [2018]):Let (S,Δ,P) be a probability space and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events Δ and a given probability function $P:\ \Delta\rightarrow\ \mathbb{R}$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-statistically convergent of order γ where($0\ <\gamma\ \leq 1$) in r^{th} expectation to a random variable X (where $X:\ S\ \rightarrow\ \mathbb{R}$) if for any $\varepsilon>0$

$\lim_{n\to\infty}\frac{1}{(\beta_n-\alpha_n+1)^{\gamma}}\,|\{k\in[\alpha_n,\beta_n]: E\,(|X_k-X|^r\geq\varepsilon\}|=0,$

Provided $E(|X_n|^r)$ and $E(|X|^r)$ exists for all $n\ \in\ \mathbb{N}$in this case we write $S_{\alpha\beta}^{r}-\lim E(|X_n-X|^r)\ =\ 0$ or by$X_n\xrightarrow{ES_{\alpha\beta}^{\gamma,r}}X$. The class of all sequences of randomvariables which are $\alpha\beta$- statistically convergent of orderγ in r^{th}expectation is denotedsimply by$ES_{\alpha\beta}^{\gamma,r}$.

Definition 2.4 (Das *et al.*, [2018]):Let (S,Δ,P) be a probability space and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events Δ and a given probability function$P:\ \Delta\rightarrow\ \mathbb{R}$. Let $F_n(x)$ be thedistribution function of X_n for all $n\ \in\mathbb{N}$. If there exist a random variable X whose distribution function is $F(x)$ such that the sequence $\{F_n(x)\}_{n\epsilon\mathbb{N}}$is $\alpha\beta-$ statistically convergent of order γ to $F(x)$ at every point of continuity x of $F(x)$ then $\{X_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta-$ statistically convergent of order γin distribution to X and we write $X_n\xrightarrow{\Delta S_{\alpha\beta}^{\gamma}}X$.

Theorem 2.1 (Das *et al.*, [2018]):If a sequence of constants $x_n\xrightarrow{S_{\alpha\beta}^{\gamma}}x$ then regarding a constant as a random variable having one point distribution at that point, we may also write $x_n\xrightarrow{\mathrm{P}\,S_{\alpha\beta}^{\gamma}}x$.

Theorem 2.2 (Elementary properties (Das *et al.*, [2018])):We have the following

i. If $X_n\xrightarrow{\mathrm{P}\,S_{\alpha\beta}^{\gamma_1}}X$ and $X_n\xrightarrow{\mathrm{P}\,S_{\alpha\beta}^{\gamma_2}}Y$ then P(X = Y) = 1 for any γ_1,γ_2 where $0<\gamma_1,\gamma_2\leq 1$.

ii. If $X_n\xrightarrow{\mathrm{P}\,S_{\alpha\beta}^{\gamma_1}}X$ and $Y_n\xrightarrow{\mathrm{P}\,S_{\alpha\beta}^{\gamma_2}}Y$ then (cX_n+dY_n) $\xrightarrow{\mathrm{P}\,S_{\alpha\beta}^{max\,\{\gamma_1\gamma_2\}}}(cX\ +\ dY)$ where c,d are constants and $0<\gamma_1,\gamma_2\leq 1$.

iii. Let 0$<\gamma_1\leq\gamma_2\leq 1$. Then P $S_{\alpha\beta}^{\gamma_1}\subseteq$ P $S_{\alpha\beta}^{\gamma_2}$ and this inclusion is strict whenever$\gamma_1<\gamma_2$.

iv. Let g:$\mathbb{R}\rightarrow\mathbb{R}$ be a continuous function and $0<\gamma_1\leq\gamma_2\leq 1$.if$X_n\xrightarrow{\mathrm{P}\,S_{\alpha\beta}^{\gamma_1}}X$ then $g(X_n)\xrightarrow{\mathrm{P}\,S_{\alpha\beta}^{\gamma_2}}g(X)$.

Theorem 2.3 (Das *et al.*, [2018]):Let $0\ <\ \gamma\ \leq 1$, (α,β) and (α',β') are two pairs of sequences of positive real numbers such that $[\alpha_n',\beta_n']\subseteq[\alpha,\beta]$ for all $n\ \in\ N$ and $(\beta_n-\alpha_n+1)^{\gamma}\leq\ \varepsilon(\beta_n'-\alpha_n'+1)^{\gamma}$for some $\varepsilon>0$, then we have P $S_{\alpha\beta}^{\gamma}\subseteq$ P $S_{\alpha'\beta'}^{\gamma}$.

Theorem 2.4 (Das *et al.*, [2018]):

i. Let $0\ <\gamma_1\leq\gamma_2\leq 1$, then$P\ W_{\alpha\beta}^{\gamma_1,p}$. This inclusion is strict whenever$\gamma_1<\gamma_2$.

ii. Let $0\ <\gamma\leq 1\ and\ 0\ <P<q\ <\infty$, then P $W_{\alpha\beta}^{\gamma,q}\subset$ P $W_{\alpha\beta}^{\gamma,p}$.

Theorem 2.5 (Das *et al.*, [2018]):Let$0\ <\gamma_1\leq\gamma_2\leq 1$ Then P $W_{\alpha\beta}^{\gamma_1,p}\subset$ P $S_{\alpha\beta}^{\gamma_2}$.

Theorem 2.6 (Das *et al.*, [2018]):
i. For $\gamma = 1$, P $W_{\alpha\beta}^{1,p} = PS_{\alpha\beta}$.
ii. Let g:$\mathbb{R} \to \mathbb{R}$ be a continuous function and $0 < \gamma_1 \le \gamma_2 \le 1$ If $x_n \xrightarrow{P\, W_{\alpha\beta}^{\gamma 1}} x$, then $g(x_n) \xrightarrow{P\, W_{\alpha\beta}^{\gamma 2}} g(x)$.
Theorem 2.7 (Das *et al.*, [2018]): Let $\{\alpha_n\}_{n\epsilon N}$,$\{\beta_n\}_{n\epsilon N}$ be two non – decreasing sequences of positive numbers such that $\alpha_n \le \beta_n \le \alpha_{n+1} \le \beta_{n+1}$ and $0 < \gamma_1 \le \gamma_2 \le 1$. Then $PS^{\gamma_1} \subset PS_{\alpha\beta}^{\gamma_2}$ iff $\liminf \left(\frac{\beta_n}{\alpha_n}\right) > 1$. (such a pair of sequence exists: take $\alpha_n = n!$ and $\beta_n = (n+1)!$.)
Theorem 2.8 (Das *et al.*, [2018]): Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$, $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ be two non – decreasing sequences of positive numbers. Let $X_n \xrightarrow{P\, S^{\gamma 1}} X$ and $X_n \xrightarrow{P\, S_{\alpha\beta}^{\gamma 2}} Y$ for $0 < \gamma_1 \le \gamma_2 \le 1$ if $\liminf \left(\frac{\beta_n}{\alpha_n}\right) > 1$ then $P\{X = Y\} = 1$.
Theorem 2.9 (Das *et al.*, [2018]): Let $X_n \xrightarrow{E\, S_{\alpha\beta}^{\gamma,r}} X$ (for any $r > 0$ and $0 < \gamma \le 1$). Then $X_n \xrightarrow{P\, S_{\alpha\beta}^{\gamma}} X$, i.e $\alpha\beta$-statistical convergence of order γ in r^{th} expectation implies $\alpha\beta-$ statistical convergence of order γ in probability.
Theorem 2.10 (Das *et al.*, [2018]): Let $\{X_n\}_{n\epsilon N}$ be a sequence of random variables such that $P(|X_n| \le M) = 1$ for all n and some constant $M > 0$. Suppose that $X_n \xrightarrow{E\, S_{\alpha\beta}^{\gamma}} X$. Then $X_n \xrightarrow{E\, S_{\alpha\beta}^{\gamma,r}} X$ for any $r > 0$.
Theorem 2.11 (Das *et al.*, [2018]):
(i) Let $X_n \xrightarrow{E\, S_{\alpha\beta}^{\gamma,r}} X$ *and* $X_n \xrightarrow{E\, S_{\alpha\beta}^{\gamma,r}} Y$(*for all* $r > 0$ *and* $0 < \gamma \le 1$). Then $P(X = Y) = 1$ provided $(X - X_n) \ge 0$ *and* $(Y_n - Y) \ge 0$.
(ii) Let $X_n \xrightarrow{E\, S_{\alpha\beta}^{\gamma,r}} X$ *and* $Y_n \xrightarrow{E\, S_{\alpha\beta}^{\gamma,r}} Y$(*for all* $r > 0$ *and* $0 < \gamma \le 1$). Then $(X_n + Y_n) \xrightarrow{E\, S_{\alpha\beta}^{\gamma}} (X + Y)$provided$(X - X_n) \ge 0$ *and* $(Y_n - Y) \ge 0$.
Theorem 2.12 (Das *et al.*, [2018]): Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Also let $f_n(x) = P(X_n = x)$ be the probability mass function of X_n for all n$\in \mathbb{N}$ and $f(x) = P(X = x)$ be the probability mass function of X. if $f_n(x) \xrightarrow{S_{\alpha\beta}^{\gamma}} f(x)$ for all x then $X_n \xrightarrow{\Delta\, S_{\alpha\beta}^{\gamma}} X$.
Theorem 2.13 (Das *et al.*, [2018]): Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. If $X_n \xrightarrow{P\, S_{\alpha\beta}^{\gamma}} X$ then $X_n \xrightarrow{\Delta\, S_{\alpha\beta}^{\gamma}} X$.that is $\alpha\beta$ – statistical convergence of order γ in probability implies $\alpha\beta-$ statistical convergence of order γ in distribution.
Theorem 2.14 (Das *et al.*, [2018]): Let $\{\alpha_n\}_{n\epsilon N}$ and $\{\beta_n\}_{n\epsilon N}$ be two increasing sequences of positive real numbers such that $\alpha_n \le \beta_n$, $\alpha_{n+1} \le \beta_{n+1}$, $(\beta_n - \alpha_n) \to \infty$ as $n \to \infty$ and $0 < \gamma_1 \le \gamma_2 \le 1$. If $\liminf \left(\frac{\beta_n}{\alpha_n}\right) > 1$ then $X \xrightarrow{P\, S^{\gamma 1}} X$ implies $X_n \xrightarrow{P\, S_{\alpha\beta}^{\gamma 2}} X$.

III. Main Results

We now give our definitions and results as follows:
Definition 3.1. Let (S_2, Δ_2, P) be a probability space and $\{X_{mn}\}_{m,n\in\mathbb{N}}$ be a double sequence of random variables where each X_{mn} is defined on the same sample space S_2(for each m, n) with respect to given class of events Δ_2 and a given probability function P: $\Delta_2 \to \mathbb{R}^2$. Then the double sequence $\{X_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $\alpha\beta -$ statistical convergence of double sequence of order $\gamma$where $(0 < \gamma \le 1)$ in probability to a random variable$X$ (*where* $X: S \times S \to \mathbb{R}^2$) if for any $\varepsilon, \delta > 0$
$\lim_{m,n\to\infty} \frac{1}{(\beta_{m,n} - \alpha_{m,n} + 1)^{\gamma}} |\{(j,k) \in [\alpha_{m,n}, \beta_{m,n}] : P(|X_{jk} - X|) \ge \varepsilon) \ge \delta\}| = 0$
Or equivalently,
$\lim_{m,n\to\infty} \frac{1}{(\beta_{m,n} - \alpha_{m,n} + 1)^{\gamma}} |\{(j,k) \in [\alpha_{m,n}, \beta_{m,n}] : 1 - P(|X_{jk} - X| < \varepsilon) \ge \delta\}| = 0.$
Example 3.1. Let a double sequence of random variables $\{x_{mn}\}_{m,n\in\mathbb{N}}$ be defined by

$$X_n \in \begin{cases} \{-1,1\}\ with\ probability\ \frac{1}{2},\ \ if\ m,n = p^2q^2 for\ some\ p,q \in \mathbb{N}, \\ \{0,1\}\ with\ probability\ P(x_{mn} = 0) = \left(1 - \frac{1}{mn}\right) and\ P(x_{mn} = 1) = \frac{1}{mn}, \\ if\ mn \ne p^2q^2, for\ any\ p,q \in \mathbb{N}. \end{cases}$$

Let $0 \le \varepsilon, \delta < 1$. Then we have

$$P(|x_{m,n} - 0| \ge \varepsilon) = 1, if\ mn = (p^2q^2)\ for\ some\ p,q \in \mathbb{N}$$

And

$$P(|x_{m,n} - 0| \geq \varepsilon) = \frac{1}{mn}\ if\ mn \neq p^2q^2 for\ any\ p,q \in \mathbb{N}$$

Let $\gamma = \frac{1}{2}, \alpha_{m,n} = ((m-1)^2(n-1)^2+1), \beta_{m,n} = m^2n^2$ for all $m,n \in \mathbb{N}$.then we have the inequality, $\frac{1}{\sqrt{2mn}-1}\left|\{(j,k) \in [(m-1)^2(n-1)^2+1, m^2n^2]: P(|X_{m,n} - 0| \leq \varepsilon) \geq \delta\}\right| = \left(\frac{1}{\sqrt{2mn}-1} + \frac{d}{\sqrt{2mn}-1}\right) \to 0.$ As $n \to \infty$ and where d is a finite positive integer. So $x_{m,n} \xrightarrow{P\ S_{2_{\alpha\beta}}^{\frac{1}{2}}} 0,$

But

$\frac{\sqrt{mn}-1}{\sqrt{mn}} \leq \frac{1}{\sqrt{mn}}|\{j \leq m, k \leq n: P(|x_{mn} - 0| \geq \varepsilon) \geq \delta\}|$. So the right hand side does not tend to 0. This shows that $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is not statistically convergent of order $\frac{1}{2}$ in probability to 0.

Theorem 3.1. If a double sequence of constants $x_{mn} \xrightarrow{S_{2_{\alpha\beta}}^{\gamma}} x$ then regarding a constant as a random variable having one point distribution at that point, we may also write

$x_{mn} \xrightarrow{P\ S_{2_{\alpha\beta}}^{\gamma}} x.$

Proof: Let $\varepsilon > 0$ be any arbitrarily small positive real number. Then$\lim_{m,n\to\infty} \frac{1}{(\beta_{mn} - \alpha_{mn} + 1)^{\gamma}}\left|\{(j,k)\epsilon[\alpha_{mn}, \beta_{mn}]: |x_{jk} - x| \geq \varepsilon\}\right| = 0$. Now let $\delta > 0$.

So the set $K_{11} = \{(j,k) \in \mathbb{N}\times\mathbb{N}: P(|x_{jk} - x| \geq \varepsilon) \geq \delta\} \subseteq K$

Where $K = \{(j,k) \in \mathbb{N}\times\mathbb{N}: |x_{jk} - x| \geq \varepsilon\}$. This shows that $x_{mn} \xrightarrow{S_{2_{\alpha\beta}}^{\gamma}} x.$

Example 3.2

Let c be a rational number between γ_1 and γ_2.Let probability density function of X_{mn} be given by

$$f_{mn}(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & otherwise \end{cases}$$

$$If\ mn = \left[(pq)^{\frac{1}{c}}\right], for\ any\ p,q \in \mathbb{N}, and$$

$$f_{mn}(x) = \begin{cases} \dfrac{mnx^{(m-1)(n-1)}}{2^{mn}}, & where\ 0 < x < 2 \\ 0, & otherwiswe \end{cases}$$

$If\ mn \neq \left[(pq)^{\frac{1}{c}}\right], for\ any\ p,q \in \mathbb{N}$. Now let $0 < \varepsilon\ and\ \delta < 1$.then

$P(|X_{mn} - 2| \geq \varepsilon) = 1, if\ mn = \left[(pq)^{\frac{1}{c}}\right]$For some $p,q \in \mathbb{N}$

And

$$P(|X_{mn} - 2| \geq \varepsilon) = \left(1 - \frac{\varepsilon}{2}\right)^{mn}, if\ mn \neq \left[(pq)^{\frac{1}{c}}\right]\ for\ any\ p,q\ \epsilon\mathbb{N}.$$

Now let $\alpha_{mn} = 1\ and\ \beta_{mn} = p^2q^2$.consequently we have the inequality

$$\lim_{mn\to\infty} \frac{(mn)^{2c} - 1}{(mn)^{2\gamma_1}} \leq \lim_{mn\to\infty} \frac{1}{(mn)^{2\gamma_1}}\left|\{(j,k) \in [1, m^2n^2]: P(|X_{jk} - 2| \geq \varepsilon) \geq \delta\}\right|$$

And

$$\lim_{mn\to\infty} \frac{1}{(mn)^{2\gamma_2}}\left|\{(j,k) \in [1, (mn)^2]: P(|X_{jk} - 2| \geq \varepsilon) \geq \delta\}\right| \leq \lim_{mn\to\infty}\left(\frac{(mn)^{2c} + 1}{(mn)^{2\gamma_2}} + \frac{d}{(mn)^{2\gamma_2}}\right),$$

Where d is a fixed finite positive integer. This shows that $\{X_{mn}\}_{m,n\in\mathbb{N}}$ is $\alpha\beta -$ statistically convergent of order γ_2 in probability to 2 but is not $\alpha\beta -$ statistically convergent of double sequence of order γ_1 in probability to 2 whenever $\gamma_1 < \gamma_2$ and this is not the usual $\alpha\beta -$ statistically convergent of double sequence of order γ of real numbers. So the converse of theorem 3.1 is not true. Also by taking $\gamma_2 = 1$, we see that $X_{mn} \xrightarrow{P\ S_{2_{\alpha\beta}}} 2$ but $\{X_{mn}\}_{m,n\in\mathbb{N}}$ is not $\alpha\beta -$ statistically convergent of double sequence of order γ in probability to 2 for $0 < \gamma < 1$.

Theorem 3.2 (Elementary properties).We have the following

i. If $X_{mn} \xrightarrow{P\ S_{2_{\alpha\beta}}^{\gamma 1}} X$ and $X_{mn} \xrightarrow{P\ S_{2_{\alpha\beta}}^{\gamma 2}} Y$ then $P\{X = Y\} = 1$ for any γ_1, γ_2 where $0 < \gamma_1, \gamma_2 \leq 1$.

ii. If $X_{mn} \xrightarrow{P\ S_{2_{\alpha\beta}}^{\gamma 1}} X$ and $Y_{mn} \xrightarrow{P\ S_{2_{\alpha\beta}}^{\gamma 2}} Y$ then $(cX_{mn} + dY_{mn}) \xrightarrow{P\ S_{2_{\alpha\beta}}^{max\ \{\gamma 1 \gamma 2\}}} (cX + dY)$ where c,d are constants and $0 < \gamma_1, \gamma_2 \leq 1$.

iii. Let $0 < \gamma_1 \leq \gamma_2 \leq 1$. Then $P\ S_{2_{\alpha\beta}}^{\gamma_1} \subseteq P\ S_{2_{\alpha\beta}}^{\gamma_2}$ and this inclusion is strict whenever $\gamma_1 < \gamma_2$.

iv. Let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ be a continuous function and $0<\gamma_1 \le \gamma_2 \le 1$. If $X_{mn} \xrightarrow{P\, S_{2_{\alpha\beta}}^{\gamma 1}} X$ then $g(X_{mn}) \xrightarrow{P\, S_{2_{\alpha\beta}}^{\gamma 2}} g(X)$.

Proof:

(i) Without loss of generality, we assume, $\gamma_2 < \gamma_1$. If possible let $P\{X = Y\} \neq 1$, then there exist two positive real numbers ε and δ such that $P(|X - Y| \ge \varepsilon) = \delta > 0$.

Then we have

$\lim_{mn\to\infty} \frac{\beta_{mn}-\alpha_{mn}+1}{(\beta_{mn}-\alpha_{mn}+1)^{\gamma 1}} \le \lim_{mn\to\infty} \frac{1}{(\beta_{mn}-\alpha_{mn}+1)^{\gamma 1}} \left|\left\{(j,k)\in[\alpha_{mn},\beta_{mn}]: P\left(\left|X_{jk}-X\right| \ge \frac{\varepsilon}{2}\right) \ge \frac{\delta}{2}\right\}\right| +$ lim*mn→∞1(βmn−αmn+1)γ2|j,k∈αmn,βmn:PXjk−Y≥ε2≥δ2|* Which is impossible because the left hand limit is not 0 whereas the right hand limit is 0.So $P\{X = Y\} = 1$.

(ii) Proof is straight forward and so is omitted.

(iii) The first part is obvious. The inclusion is proper as can be seen from Example 3.2.

(iv) Proof is straight forward and so is omitted.

Theorem 3.3. Let $0 < \gamma \le 1$, (α,β) and (α',β') are two pairs of double sequences of positive real numbers such that $[\alpha'_{mn},\beta'_{mn}] \subseteq [\alpha,\beta]$ for all $mn\,\epsilon\mathbb{N}$ and $(\beta_{mn}-\alpha_{mn}+1)^{\gamma} \le \varepsilon(\beta'_{mn}-\alpha'_{mn}+1)^{\gamma}$ for some $\varepsilon>0$, then we have $P\, S_{2_{\alpha,\beta}}^{\gamma} \subseteq P\, S_{2_{\alpha',\beta'}}^{\gamma}$.

Proof: Is straight forward and is omitted. But if the condition of theorem 3.3 is violated then the limit may not be unique for two different $(\alpha,\beta)'s$we now give an example to show this.

Example 3.3. Let $\alpha = \{(2mn)!\}, \beta = \{(2m+1)!\,(2n+1)!\}and\alpha'$=$\{(2m+1)!\,(2n+1)!\}$. Let us define a sequence of random variables $\{X_{mn}\}_{mn\in\mathbb{Z}}$ by

$$X_{jk} \in \begin{cases} \{-1,1\}\ with\ probability\ P(X_{jk}=-1) = \dfrac{1}{jk}, P(x_{jk}=1) = \left(1-\dfrac{1}{jk}\right), \\ \quad if\ (2m)! < j < (2m+1)!, (2n)! < k < (2n+1)!, \\ \\ \{-2,2\}\ with\ probability\ P(x_{jk}=-2) = \dfrac{1}{jk},\ \ P(x_{jk}=2) = \left(1-\dfrac{1}{jk}\right), \\ \quad if\ (2m+1)! < j < (2m+2)!, (2n+1)! < k < (2n+2)!, \\ \\ \{-3,3\}\ with\ probability\ P(x_{jk}=-3) = \dfrac{1}{k},\ \ P(x_{jk}=3), \\ \quad if\ j=(2m)!\ and\ j=(2m+1)!, if\ k=(2n)!\ and\ k=(2n+1)! \end{cases}$$

Let $0<\varepsilon,\delta<1$ and $0<\gamma<1$. Then for the double sequence (α,β)

$$P(|x_{jk}-1| \ge \varepsilon) = \frac{1}{jk}, if\ (2m)! < j < (2m+1)!, (2n)! < k < (2n+1)!,$$

And

$$P(|x_{jk}-1| \ge \varepsilon) = 1, if\ (2m+1)! < j < (2m+2)!, (2n+1)! < k < (2n+2)!,$$

And

$$P(|x_{jk}-1| \ge \varepsilon) = 1, if\ j=(2m)!\ and\ j=(2m+1)!, k=(2n)!\ and\ k=(2n+1)!,$$

Therefore,

$$\lim_{mn\to\infty} \frac{1}{((2m+1)!\,(2n+1)! - (2m)!\,(2n)! + 1)^{\gamma}} |\{j\epsilon[(2m)!,(2m+1)!], k\epsilon[(2n)!,(2n+1)!,]: P(|x_k - 1| \ge \varepsilon) \ge \delta\}| = 0$$

So $x_{mn} \xrightarrow{PS_{2_{\alpha\beta}}^{\gamma}} 1$. Similarly, it can be shown that for all the double sequences $\alpha' = \{(2m+1)!\,(2n+1)!\}, \beta' = \{(2m+2)!\,(2n+2)!\}$, and $x_{mn} \xrightarrow{PS_{2_{\alpha',\beta'}}^{\gamma}} 2$.

Definition 3.2. Let (S_2,Δ_2,P) be a probability space and $\{X_{mn}\}_{m,n\in\mathbb{N}}$be a double sequence of random variables where each X_{mn} is defined on the same sample space S_2 (for each m and n) with respect to a given class of events Δ_2 and a given probability function$P:\Delta_2 \to \mathbb{R}^2$. A double sequence of random variables$\{X_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $\alpha\beta$ −strong p-Cesàro summable of order γ where ($0 <\gamma \le 1$) and p > 0 is any fixed positive real number) in probability to a random variable X if for any $\varepsilon > 0$

$$\lim_{mn\to\infty} \frac{1}{(\beta_{mn}-\alpha_{mn}+1)^{\gamma}} \sum_{j\in[\alpha_m,\beta_m]} \sum_{k\in[\alpha_n,\beta_n]} \{P(|X_{jk}-X| \ge \varepsilon)^p\} = 0.$$

In this case we write $X_{mn} \xrightarrow{\mathrm{p}W_{2_{\alpha,\beta}}^{\gamma,p}} X$. The class of all sequences of random variables which are $\alpha\beta$-strong p-Cesàro summable of order γ in probability is denoted simplyby $PW_{2_{\alpha,\beta}}^{\gamma,p}$.

Theorem 3.4.

i. Let $0 < \gamma_1 \leq \gamma_2 \leq 1$, thenP$W_{2_{\alpha,\beta}}^{\gamma_1,p}$. This inclusion is strict whenever $\gamma_1 < \gamma_2$.

ii. Let $0 < \gamma \leq 1$ and $0 < P < q < \infty$, then P $W_{2_{\alpha,\beta}}^{\gamma,q} \subset$ P $W_{2_{\alpha,\beta}}^{\gamma,p}$.

Proof:

(i) The first part of this theorem is straightforward and so is omitted. For the second part we will give an example to show that there is a double sequence of random variables $\{X_{mn}\}_{m,n\in\mathbb{N}}$ which is $\alpha\beta$ −strong p-Cesàro summable of orderγ_2 of double sequence in probability to a random variable X but is not $\alpha\beta$ −strong p-Cesàro summable of orderγ_1 of double sequence in probability whenever $\gamma_1 < \gamma_2$.
Let c be a rational number between γ_1 and γ_2. We consider the sequence of random Variables:

$$X_{mn} \in \begin{cases} \{-1,1\}\ with\ probability\ \dfrac{1}{2}, if\ mn = \left[(pq)^{\frac{1}{c}}\right]\ for\ some\ m,n \in \mathbb{N} \\ \{0,1\}\ with\ probability\ P(X_{mn} = 0) = 1 - \dfrac{1}{\sqrt[P]{m^2n^2}}\ and\ P(X_{mn} = 1) = \dfrac{1}{\sqrt[P]{m^2n^2}} \\ if\ mn \neq \left[(pq)^{\frac{1}{c}}\right]\ for\ any\ m,n \in \mathbb{N} \end{cases}$$

Then we have for $0 < \varepsilon < 1, P(|X_{mn} - 0| \geq \varepsilon) = 1, if\ m,n = \left[(pq)^{\frac{1}{c}}\right]$for some $m,n \in \mathbb{N}$

And $P(|X_{mn} - 0| \geq \varepsilon) = \frac{1}{\sqrt[P]{m^2n^2}}, if\ m,n = \left[(pq)^{\frac{1}{c}}\right] for\ any\ m,n\epsilon\mathbb{N}$.

Let $\alpha_{mn} = 1$ and $\beta_{mn} = p^2q^2$, so we have the inequality

$$\lim_{m,n\to\infty} \frac{(mn)^{2c} - 1}{(mn)^{2\gamma_1}} \leq \lim_{m,n\to\infty} \frac{1}{(mn)^{2\gamma_2}} \sum\nolimits_{j\epsilon\,[1,m^2]} \sum\nolimits_{k\epsilon\,[1,n^2]} \{P(|X_{jk} - 0| \geq \varepsilon)\}^p$$

And

$$\lim_{m,n\to\infty} \frac{1}{(mn)^{2\gamma_2}} \sum\nolimits_{j\epsilon\,[1,m^2]} \sum\nolimits_{k\epsilon\,[1,n^2]} \{P(|X_{jk} - 0| \geq \varepsilon)\}^p \leq \lim_{m,n\to\infty} \left[\frac{(mn)^{2c} + 1}{(mn)^{2\gamma_2}} + \frac{1}{(mn)^{2\gamma_2}}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(mn)^4}\right)\right]$$

This shows that $X_{mn} \xrightarrow{\mathrm{P}\,W_{2_{\alpha\beta}}^{\gamma 2,p}} 0$ but $\{X_{mn}\}_{m,n\in\mathbb{N}}$ is not $\alpha\beta$ −strong p-Cesàro summable of orderγ_1 of double sequence in probability to 0.

(ii) Proof is straight forward and so is omitted.

Theorem 3.5

Let $0 < \gamma_1 < \gamma_2 < 1$. Then P $W_{2_{\alpha\beta}}^{\gamma_1,p} \subset$ P $S_{2_{\alpha\beta}}^{\gamma_2}$.

Proof: proof is straight forward and so is omitted.

So we can say that if a double sequence of random variables $\{X_{mn}\}_{m,n\in\mathbb{N}}\alpha\beta$ −strong p-Cesàro summable of orderγ of double sequence in probability to X then it is $\alpha\beta$ −statistically convergent of double sequences in probability to X i.e.P $W_{2_{\alpha\beta}}^{\gamma,p} \subset$ P $S_{2_{\alpha\beta}}^{\gamma}$.

But the converse of the theorem 3.5 is not true in general as can be seen from the following example.

Example 3.4

Let a double sequence of random variables $\{X_{mn}\}_{m,n\in\mathbb{N}}$ be defined by

$$X_{mn} \in \begin{cases} \{-1,1\}\ with\ probability\ \dfrac{1}{2}, if\ mn = (pq)^{pq} for\ some\ p,q\epsilon\mathbb{N}, \\ \{0,1\}\ with\ probability\ P(X_{mn} = 0) = 1 - \dfrac{1}{\sqrt[2p]{mn}}\ and\ P(X_{mn} = 1) = \dfrac{1}{\sqrt[2p]{mn}}, \\ if\ mn \neq (pq)^{pq} for\ any\ p,q \in \mathbb{N}. \end{cases}$$

Let $0 < \varepsilon < 1$ be given. Then

$$P(|X_{mn} - 0| \geq \varepsilon) = 1, if\ mn = (pq)^{pq} for\ some\ p,q\ \epsilon\mathbb{N}$$

And

$P(|X_{mn} - 0| \geq \varepsilon) = \frac{1}{\sqrt[2p]{mn}}, if\ mn \neq (pq)^{pq} for\ any\ (p,q)\ \epsilon\mathbb{N}$.

Let $\alpha_{mn} = 1$ and $\beta_{mn} = (mn)^2$. This implies $X_{mn} \xrightarrow{\mathrm{P}\,W_{2_{\alpha\beta}}^{\gamma}} 0$ for each $0 < \gamma \leq 1$. Next let $H = \{m,n \in \mathbb{N}: mn \neq (pq)^{pq} for\ any\ p,q \in \mathbb{N}\}$. Then

$$\frac{1}{(mn)^{2\gamma}}\sum_{j\epsilon[1,m^2]}\sum_{k\epsilon[1,n^2]}\{P(|X_{jk}-0|\geq\varepsilon)\}^p$$
$$=\frac{1}{(mn)^{2\gamma}}\sum_{\substack{j\epsilon[1,m^2]\\ j\in H}}\sum_{\substack{k\epsilon[1,n^2]\\ k\in H}}\{P(|X_{jk}-0|\geq\varepsilon)\}^p$$
$$+\frac{1}{(mn)^{2\gamma}}\sum_{\substack{j\epsilon[1,m^2]\\ j\notin H}}\sum_{\substack{k\epsilon[1,n^2]\\ k\notin H}}\{P(|X_{jk}-0|\geq\varepsilon)\}^p$$
$$=\frac{1}{(mn)^{2\gamma}}\sum_{\substack{j\epsilon[1,m^2]\\ j\in H}}\sum_{\substack{k\epsilon[1,n^2]\\ k\in H}}\frac{1}{\sqrt{JK}}+\frac{1}{(mn)^{2\gamma}}\sum_{\substack{j\epsilon[1,m^2]\\ j\notin H}}\sum_{\substack{k\epsilon[1,n^2]\\ k\notin H}}1>\frac{1}{(mn)^{2\gamma}}\sum_{j=1}^{m^2}\sum_{k=1}^{n^2}\frac{1}{\sqrt{JK}}>\frac{1}{(mn)^{2\gamma-1}}$$

Since $\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{\sqrt{jk}}>\sqrt{mn},\ for\ m,n\geq 2$ so X_{mn} is not $\alpha\beta-$ strong p-cesaro summable of double sequence of order Y in probability to 0 for $0<\gamma\leq\frac{1}{2}$.

Theorem 3.6

(i) For $\gamma=1,\ pW_{2_{\alpha\beta}}^{1,p}=PS_{2_{\alpha\beta}}$

(ii) Let $g:\mathbb{R}\times\mathbb{R}\to R\times R$ be a continuous function and $0<\gamma_1\leq\gamma_2\leq 1$

$if\ X_{mn}\xrightarrow{PW_{2_{\alpha\beta}}^{\gamma 1}}X$ then $g(X_{mn})\xrightarrow{PW_{2_{\alpha\beta}}^{\gamma 2}}g(X)$

Proof: for (i) and (ii) the proof is straight forward and so is omitted.

Theorem 3.7

Let $\{\alpha_{mn}\}_{m,n\epsilon N}$, $\{\beta_{mn}\}_{m,n\epsilon N}$ be two non – decreasing double sequences of positive numbers such that $\alpha_{mn}\leq\beta_{mn}\leq\alpha_{mn+1}\leq\beta_{mn+1}$ and $0<\gamma_1\leq\gamma_2\leq 1$. Then $\mathrm{PS}_2^{\gamma_1}\subset\mathrm{PS}_{2_{\alpha\beta}}^{\gamma_2}\ iff\ \lim inf\left(\frac{\beta_{mn}}{\alpha_{mn}}\right)>1$. (such a pair of double sequence exists: take $\alpha_{mn}=(mn)!$ and $\beta_{mn}=((m+1)!\,(n+1)!)$

Proof: First of all suppose $\lim inf\left(\frac{\beta_{mn}}{\alpha_{mn}}\right)>1$ and let $X_{mn}\xrightarrow{PS_2^{\gamma 1}}X$.as$lim inf\left(\frac{\beta_{mn}}{\alpha_{mn}}\right)>1$, for each $\delta>0$ we can find sufficiently large r and s such that $\frac{\beta_{r,s}}{\alpha_{r,s}}\geq(1+\delta)$

Therefore,

$\left(\frac{\beta_{r,s}-\alpha_{r,s}}{\beta_{r,s}}\right)^{\gamma_1}\geq\left(\frac{\delta}{1+\delta}\right)^{\gamma_1}$Now for each $\varepsilon,\delta>0$ we have

$$\frac{1}{[\beta_{mn}]^{\gamma_1}}\left|\{j\leq[\beta_m],k\leq[\beta_n]:P(|X_{jk}-X|\geq\varepsilon)\geq\delta\}\right|$$
$$=\frac{1}{[\beta_{mn}]^{\gamma_1}}\left|\{j\leq\beta_m,k\leq\beta_n:P(|X_{jk}-X|\geq\varepsilon)\geq\delta\}\right|$$
$$\geq\frac{1}{\beta_{mn}^{\gamma_1}}\left|\{j\leq\beta_m,k\leq\beta_n:P(|X_{jk}-X|\geq\varepsilon)\geq\delta\}\right|$$
$$\geq\left(\frac{\delta}{1+\delta}\right)^{\gamma_1}\frac{1}{(\beta_{mn}-\alpha_{mn}+1)^{\gamma_2}}\left|\{j\epsilon[\alpha_{m,}\beta_m],k\epsilon[\alpha_{n,}\beta_n]:P(|X_{jk}-X|\geq\varepsilon)\geq\delta\}\right|.$$

Hence the result follows: now if possible, suppose that $\lim inf\left(\frac{\beta_{mn}}{\alpha_{mn}}\right)>1$so for each $j,k\in\mathbb{N}$, we can choose a subsequence such that $\frac{\beta_{r_j,s_k}}{\alpha_{r_j,s_k}}<1+\frac{1}{jk}\ and\ \beta_{r_j-1,s_{k-1}}\geq jk$.

Let $I_{r_j,s_k}=\left[\alpha_{r_j,s_k},\beta_{r_j,s_k}\right]$. We define a double sequence of random variables by

$$X_{mn}\in\begin{cases}\left\{-1,1\ with\ probability\ \frac{1}{2},if\ m,n\in\ I_{r_j,s_k}where\ j,k\in\mathbb{N}\right\}\\ \{0,1\}\ with\ probability\ P(X_{mn}=0)=1-\frac{1}{(mn)^2}\ and\ P(X_{mn}=1)=\frac{1}{(mn)^2}\\ if\ m,n\notin I_{r_j,s_k}\ for\ any\ j,k\,\epsilon\,\mathbb{N}\end{cases}$$

Let $0<\varepsilon,\delta>1$. Now

$$P(|X_{mn}-0|\geq\varepsilon)=1,if\ m,n\in I_{r_j,s_k}\ where\ j,k\,\epsilon\,\mathbb{N},$$

And

$$P(|X_{mn}-0|\geq\varepsilon)=\frac{1}{(mn)^2},if\ m,n\notin I_{r_j,s_k}\ where\ j,k\,\epsilon\,\mathbb{N},$$

Now $\frac{1}{\left(\beta_{r_j,s_k}-\alpha_{r_j,s_k}+1\right)^{\gamma}}\left|\left\{j,k \in \left[\alpha_{r_j,s_k},\beta_{r_j,s_k}\right]: P(|X_{mn}-0| \geq \varepsilon) \geq \delta\right\}\right| = \frac{\left(\beta_{r_j,s_k}-\alpha_{r_j,s_k}+1\right)}{\left(\beta_{r_j,s_k}-\alpha_{r_j,s_k}+1\right)^{\gamma}} \to \infty$ but as $\frac{1}{\left(\beta_{r_j,s_k}-\alpha_{r_j,s_k}+1\right)^{\gamma}}\left|\left\{j,k \in \left[\alpha_{r_j,s_k},\beta_{r_j,s_k}\right]: P(|X_{mn}-0| \geq \varepsilon) \geq \delta\right\}\right.$ is a subsequence of the double sequence$\frac{1}{\left(\beta_{r,s}-\alpha_{r,s}+1\right)^{\gamma}}\left|\left\{j,k \in \left[\alpha_{r_j,s_k},\beta_{r_j,s_k}\right]: P(|X_{mn}-0| \geq \varepsilon) \geq \delta\right\}\right.$ this shows that X_{mn} is not $\alpha\beta-$ statistically convergent of double sequences of order γ (where 0< $\gamma \leq 1$) in probability to zero.
Finally, let $\gamma = 1$. If we take u and v sufficiently large such that $\alpha_{rj} < u \leq \beta_{rj}$ and $\alpha_{sk} < v \leq \beta_{sk}$ then we observe that

$$\frac{1}{uv}\sum_{j=1}^{u}\sum_{k=1}^{v} p(|x_{jk}-0| \geq \varepsilon) \leq \frac{\beta_{r_{j-1},s_{k-1}}+\beta_{r_j,s_k}-\alpha_{r_j,s_k}}{\alpha_{r_j,s_k}} + \frac{1}{u}\left\{1+\frac{1}{(2,2)^2}+\cdots+\frac{1}{(u,v)^2}\right\}$$

$\leq \frac{\beta_{r_{(j-1)},s_{(k-1)}}}{\beta_{r_{(j)-1},s_{(k)-1}}}+\frac{\beta_{r_j,s_k}-\alpha_{r_j,s_k}}{\alpha_{r_j,s_k}}+\frac{1}{uv}\left\{1+\frac{1}{(2,2)^2}+\cdots+\frac{1}{(u,v)^2}\right\}$

$\leq \frac{2}{j,k}+\frac{1}{uv}\left\{1+\frac{1}{(2,2)^2}+\cdots+\frac{1}{(u,v)^2}\right\} \to 0, if\ j,u \to \infty\ and\ k,v \to \infty$This shows that

$X_{mn} \xrightarrow{PS_2} 0$ But this is a contradiction as $\mathrm{P}S_2^{\gamma_1} \subset \mathrm{P}S_{2_{\alpha\beta}}^{\gamma_2}$ where (0< $\gamma_1 \leq \gamma_2 \leq 1$).we conclude that lim inf$\left(\frac{\beta_{mn}}{\alpha_{mn}}\right)$ must be > 1.

Theorem 3.8

Let $\alpha = \{\alpha_{mn}\}_{m,n\epsilon\mathbb{N}}$, $\beta = \{\beta_{mn}\}_{m,n\epsilon\mathbb{N}}$ be two non-decreasing double sequences of positive numbers. Let $X_{mn} \xrightarrow{PS_2^{\gamma 1}} X$ and $X_{mn} \xrightarrow{PS_{2_{\alpha\beta}}^{\gamma 2}} \gamma$ for $0 < \gamma_2 \leq \gamma_1 \leq 1$. If $lim\ \inf\left(\frac{\beta_{mn}}{\alpha_{mn}}\right) \geq 1$, then $P\{X = Y\}$=1.

Proof:

Let $\varepsilon > 0$ be any small positive real number and if possible let $P(|X-Y| \geq \varepsilon)$=$\delta > 0$ now, we have the inequality

$$P(|X-Y| \geq \varepsilon) \leq \left\{P(|X_{mn}-X| \geq \frac{\varepsilon}{2}\right\} + \left\{P(|X_{mn}-Y| \geq \frac{\varepsilon}{2}\right\}$$

$$\Longrightarrow \left\{j,k\epsilon\left[\alpha_{mn,}\beta_{mn}\right]: P(|X-Y| \geq \delta\right\}$$

$$\subseteq \left\{j,k\epsilon[\alpha_{mn},\beta_{mn}]: P(\left|X_{jk}-X\right|) \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\right\} \cup \left\{j,k\epsilon[\alpha_{mn},\beta_{mn}]: P(|X_{jk}-Y| \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\right\}$$

$$\Longrightarrow \left|\left\{j,k\epsilon\left[\alpha_{mn,}\beta_{mn}\right]: P(|X-Y| \geq \varepsilon\right\}\right|$$

$$\leq \left|\left\{j,k\epsilon[\alpha_{mn},\beta_{mn}]: P(\left|X_{jk}-X\right|) \geq \frac{\varepsilon}{2}) \geq \frac{\delta}{2}\right\}\right| + \left|\left\{j,k\epsilon[\alpha_{mn},\beta_{mn}]: P\left(\left|X_{jk}-Y\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|$$

$$\Longrightarrow \left|\left\{j,k\epsilon\left[\alpha_{mn,}\beta_{mn}\right]: P(|X-Y| \geq \varepsilon) \geq \delta\right\}\right|$$

$$\leq \left|\left\{j \leq [\beta_m], k \leq [\beta_n]: P\left(\left|X_{jk}-X\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|$$

$$+ \left|\left\{j,k\epsilon[\alpha_{mn},\beta_{mn}]: P\left(\left|X_{jk}-Y\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|$$

$\Longrightarrow (\beta_{mn}-\alpha_{mn}) \leq \left|\left\{j \leq [\beta_m], k \leq [\beta_n]: P\left(\left|X_{jk}-Y\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| + \left|\left\{j,k\epsilon[\alpha_{mn},\beta_{mn}]: P(|X_{jk}-Y| \geq \frac{\varepsilon}{2}) \geq \right.\right.$ $\delta 2|$

$$\Longrightarrow \left(\frac{\beta_{mn}-\alpha_{mn}}{\beta_{mn}}\right)^{\gamma_1}$$

$$\leq \frac{1}{[\beta_{mn}]^{\gamma_1}}\left|\left\{j \leq [\beta_m], k \leq [\beta_n]: P\left(\left|X_{jk}-X\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|$$

$$+ \frac{1}{\beta_{mn}^{\gamma_1}}\left|\left\{j,k\epsilon[\alpha_{mn},\beta_{mn}]: P\left(\left|X_{jk}-Y\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|$$

$$\Longrightarrow \left(\frac{\beta_{mn}-\alpha_{mn}}{\beta_{mn}}\right)^{\gamma_1}$$

$$\leq \frac{1}{[\beta_{mn}]^{\gamma_1}}\left|\left\{j \leq [\beta_m], k \leq [\beta_n]: P\left(\left|X_{jk}-X\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| + \left(\frac{\beta_{mn}-\alpha_{mn}+1}{\beta_{mn}}\right)^{\gamma_2}$$

$$\times \frac{1}{(\beta_{mn}-\alpha_{mn}+1)^{\gamma_2}}\left|\left\{j,k\epsilon[\alpha_{mn},\beta_{mn}]: P\left(\left|X_{jk}-Y\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|$$

$$\Rightarrow \left(1-\frac{\alpha_{mn}}{\beta_{mn}}\right)^{\gamma_1} \leq \frac{1}{[\beta_{mn}]^{\gamma_1}}\left|\left\{j \leq [\beta_m], k \leq [\beta_n]: P\left(\left|X_{jk}-X\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| + \left(1-\frac{\alpha_{mn}}{\beta_{mn}}+\frac{1}{\beta_{mn}}\right)^{\gamma_2}$$
$$\times \frac{1}{(\beta_{mn}-\alpha_{mn}+1)^{\gamma_2}}\left|\left\{j, k \epsilon [\alpha_{mn}, \beta_{mn}]: P\left(\left|X_{jk}-Y\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|$$
$$\Rightarrow \left(1-\frac{\alpha_{mn}}{\beta mn}\right)^{\gamma_1} \leq \frac{1}{[\beta_{mn}]^{\gamma_1}}\left|\left\{j \leq [\beta_m], k \leq [\beta_n]: P\left(\left|X_{jk}-X\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| + \left(1-\frac{\alpha_{mn}}{\beta mn}+\frac{1}{\beta_{mn}}\right)^{\gamma_2}$$
$$\times \frac{1}{(\beta_{mn}-\alpha_{mn}+1)^{\gamma_2}}\left|\left\{j, k \epsilon [\alpha_{mn}, \beta_{mn}]: P\left(\left|X_{jk}-Y\right| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|.$$

Taking $m, n \longrightarrow \infty$ on both sides, we see that the left hand side does not tend to 0. Since $\liminf\left(\frac{\beta_{mn}}{\alpha_{mn}}\right) > 1$ but the right hand side tends to 0. This is a contradiction, so we must have

$$P\{X = Y\} = 1.$$

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