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# $I_2$ −Lacunary Statistical Convergence of Double Sequences of Order $\alpha$ of Sets

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***Abstract:*** *In this paper, we shall introduce the extension of recently introduced concepts of Wijsman $I$−lacunary statistical convergence of order $\alpha$ and Wijsman strongly $I$ −lacunary statistical convergenceof order $\alpha$to double sequences.*

***Keywords and phrases:*** *$I_2$ −convergence, Wijsman convergence, lacunary double sequences.*
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## I. Introduction

The concept of statistical convergence was formally introduced by Fast [8] and Schoenberg [18] independently. Although statistical convergence was introduced over fifty years ago, it has become an active area of research in recent years. It has been applied in various areas such as summability theory (Fridy [9] and Salat [16]), topological groups (Cakalli [2], [3]), topological spaces (Maio and Kocinac [11]), locally convex spaces (Maddox [12]), measure theory (Cheng et al [4]), (Connor and Swardson [5]) and (Miller [13]), Fuzzy Mathematics (Nuray and Savas [15] and Savas [17]). In recent years generalization of statistical convergence has appeared in the study of strong summability and the structure of ideals of bounded functions, (Connor and Swardson [6]).Mursaleen and Edely [14] extended the notion of statistical convergence of single sequence to double sequences. Kostyrko et al [10] further extended the idea of statistical convergence to $I$ −convergence using the notion of ideals of $\mathbb{N}$ with many interesting consequences. Das and Savas [7] introduced and studied $I$ −statistical and $I$ −lacunary statistical converegence of order $\alpha$.Quite recently Brono et al [1] introduced the concept of $I_2$ −statistical and $I_2$ −lacunary statistical convergence for double sequence of order $\alpha$ in line of Das and Savas [7]. Ulusu and Nuray [20] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence. Later on it was further studied by Seng$\ddot{u}$l and Et [19]. In this paper we shall extend these concepts to double sequences in line of Seng$\ddot{u}$l and Et [19].

If $X$ is a non-empty set then a family of set $I \subset P(X)$ is called an ideal in $X$ if and only if (i)$\Phi \in I$; $(ii)$ for each $A, B \in I$ we have $A \cup B \in I$; $(iii)$for each $A \in I$ and $B \subset A$ we have $B \in I$.

Let $X$ is a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a filter on $X$ if and only if (i) $\Phi \notin F$; $(ii)$for each $A, B \in F$ we have $A \cap B \in F$; $(iii)$for each $A \in F$ and $B \supset A$ we have $B \in F$.

An ideal $I$ is called non-trivial if $I \neq \Phi$ and $X \notin I$.

A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in $X$ if and only if it contains all singletons, i.e., if it contains $\{\{x\}: x \in X\}$.

Throughout the paper we take $I_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $I_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{x\} \times \mathbb{N}$ and $\mathbb{N} \times \{x\}$ belong to $I_2$ for each $x \in \mathbb{N}$.

For further study we shall take $X = \mathbb{N} \times \mathbb{N}$ and $I_2$ will denote an ideal of subsets of $\mathbb{N} \times \mathbb{N}$. The following proposition express a relation between the notions of an ideal and a filter.

Let $I_2 \subset P(\mathbb{N} \times \mathbb{N})$ be a non- trivial ideal. Then the class $F = F(I) = \{M \subset \mathbb{N} \times \mathbb{N}: M = \mathbb{N} \times \mathbb{N} - A, for\ some\ A \in I\}$ is a filter on $\mathbb{N} \times \mathbb{N}$ (we shall call $F = F(I)\ the\ filter\ associated\ with\ I$).

Let $(X, d)$ be a metric space. For any non-empty closed subset $A_{j,k}$ of $X$, we

Say that the double sequence $\{A_{j,k}\}$ is bounded if $\sup_{j,k} d(x, A_{j,k}) < \infty$ for each $x \in X$. In this case we write, $\{A_{j,k}\} \in L_\infty^2$.

## II. Main Results

In this section, we shallextend the results of Sengül and Et [19] to Wijsman $I$ −lacunary statistical convergence of double sequences of order $\alpha$ and Wijsman strongly $I$ −lacunary statistical convergence of double sequences of order $\alpha$ and examine some relationship between these concepts.

**Definition 2.1([19]):** Let $(X, d)$ be a metric space, $\theta$ be a lacunary sequence, $\alpha \in (0,1]$ and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets of $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $I$ −lacunary statistical convergent to $A$ of order $\alpha$($0r$ $S_\theta^\alpha(I_w) - convergent\ to\ A$)if for each $\varepsilon > 0, \delta > 0$ and $x \in X$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $I$.

**Definition 2.2([19]):** Let $(X, d)$ be a metric space, $\theta$ be a lacunary sequence, $\alpha \in (0,1]$ and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets of $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman strongly$I$ −lacunary statistical convergent to $A$ of order $\alpha$ ($N_\theta^\alpha(I_w) - convergent\ to\ A$) if for each $\varepsilon > 0, \delta > 0$ and $x \in X$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}$$

belongs to $I$.
We now give our main definitions and results.

**Definition 2.3:** Let $(X, d)$ be a metric space, $\theta_{r,s}$ be a double lacunary sequence, $0 < \alpha \leq 1$ and $I_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N} \times \mathbb{N}$. For any non-empty closed subsets of $A, A_{j,k} \subset X$, we say that the double sequence $\{A_{j,k}\}$ is Wijsman $I_2$ −lacunary statistical convergent to $A$ of order $\alpha$ $\left(S_{\theta_{r,s}}^\alpha(I_{2w}) - convergent\ to\ A\right)$ if for each $\varepsilon > 0, \delta > 0$ and $x \in X$,

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{r,s}^\alpha} \left|\left\{(j,k) \in I_{r,s}: \left|d\left(x, A_{j,k}\right) - d(x, A)\right| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

belongs to $I_2$. In this case, we write $A_{j,k} \to \left(S_{\theta_{r,s}}^\alpha(I_{2w})\right)$.
Consider the following example:

$$A_{jk} = \begin{cases} \{3x\}, & j_{r-1}k_{s-1} < jk < j_{r-1}k_{s-1} + \sqrt{h_{r,s}} \\ \{0\}, & otherwise \end{cases}$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X, d(x, y) = |x - y|, A = \{1\}, x > 1$ and $\alpha = 1$.
Since

$$\frac{1}{h_{r,s}^\alpha} \left|\left\{(j,k) \in I_{r,s}: \left|d\left(x, A_{j,k}\right) - d(x, 1)\right| \geq \varepsilon\right\}\right| \geq \delta$$

Belongs to $I_2$, the double sequence $\{A_{j,k}\}$is Wijsman $I_2$ −lacunary statistical convergent to $\{1\}$ of order $\alpha$, that is $A_{j,k} \to \{1\} \left(S_{\theta_{r,s}}^\alpha(I_{2w})\right)$.

**Definition 2.4:** Let $(X, d)$ be a metric space, $\theta_{r,s}$ be a double lacunary sequence, $0 < \alpha \leq 1$ and $I_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N} \times \mathbb{N}$. For any non-empty closed subsets of $A, A_{j,k} \subset X$, we say that the double sequence $\{A_{j,k}\}$ is Wijsman strongly $I_2$ −lacunary statistical convergent to $A$ of order $\alpha$ $\left(N_{\theta_{r,s}}^\alpha(I_{2w}) - convergent\ to\ A\right.$ if for each $\varepsilon>0,\delta>0$ and $x\in X$,

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{r,s}^\alpha} \sum_{(j,k)\in I_{r,s}} \sum \left|d\left(x, A_{j,k}\right) - d(x, A)\right| \geq \varepsilon\right\}$$

belongs to $I_2$. In this case, we write $A_{j,k} \to A \left(N_{\theta_{r,s}}^\alpha(I_{2w})\right)$.
Consider the following example:

$$A_{jk} = \begin{cases} \left\{\frac{3jk}{2}\right\}, & j_{r-1}k_{s-1} < jk < j_{r-1}k_{s-1} + \sqrt{h_{r,s}} \\ \{0\}, & otherwise \end{cases}$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X, d(x, y) = |x - y|, A = \{1\}, x > 1$ and $\alpha = 1$.
Since

$$\frac{1}{h_{r,s}^{\alpha}} \sum_{(j,k)\in I_{r,s}} \sum \left| d\left(x, A_{j,k}\right) - d(x,1) \right| \geq \varepsilon,$$

the double sequence $\{A_{j,k}\}$is Wijsman $I_2$ −lacunary statistical convergent to $\{1\}$ of order $\alpha$, that is $A_{j,k} \to \{1\} \left(N_{\theta_{r,s}}^{\alpha}(I_{2w})\right)$.

**Theorem 2.1:** Let $(X,d)$ be a metric space, $\theta_{r,s}$ be a double lacunary sequence and $A, A_{j,k}$ $(for\ all\ j,k \in \mathbb{N})$ be a non-empty closed subsets of $X$, then

(i) $A_{j,k} \to A \left(N_{\theta_{r,s}}^{\alpha}(I_{2w})\right) \Rightarrow A_{j,k} \to A \left(S_{\theta_{r,s}}^{\alpha}(I_{2w})\right)$ and $N_{\theta_{r,s}}^{\alpha}(I_{2w})$ is a proper subset of $S_{\theta_{r,s}}^{\alpha}(I_{2w})$,

(ii) $\{A_{j,k}\} \in L_{\infty}^2$ and $A_{j,k} \to A \left(S_{\theta_{r,s}}^{\alpha}(I_{2w})\right) \Rightarrow A_{j,k} \to A \left(N_{\theta_{r,s}}^{\alpha}(I_{2w})\right)$,

(iii) $S_{\theta_{r,s}}^{\alpha}(I_{2w}) \cap L_{\infty}^2 = N_{\theta_{r,s}}^{\alpha}(I_{2w}) \cap L_{\infty}^2$.

**Proof:** (i) The inclusion part of the proof is easy. In order to show that the inclusion $N_{\theta_{r,s}}^{\alpha}(I_{2w}) \subseteq S_{\theta_{r,s}}^{\alpha}(I_{2w})$ is proper, let $\theta_{r,s}$ be given and we write a double sequence $\{A_{j,k}\}$ as follows

$$A_{jk} = \begin{cases} \{x^2\}, j,k = 1,2,3, \dots \left[\sqrt{h_{r,s}}\right] \\ \{0\}, \qquad otherwise \end{cases}.$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X, d(x,y) = |x-y|$. We have for every $\varepsilon > 0, x > 0$ and $\frac{1}{2} < \alpha \leq 1$,

$$\frac{1}{h_{r,s}^{\alpha}} \left| \left\{ (j,k) \in I_{r,s} : \left| d\left(x, A_{j,k}\right) - d(x,\{0\} \right| \geq \varepsilon \right\} \right| \leq \frac{\left[\sqrt{h_{r,s}}\right]}{h_{r,s}^{\alpha}},$$

And for any $\delta > 0$ we get

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}^{\alpha}} \left| \left\{ (j,k) \in I_{r,s} : \left| d\left(x, A_{j,k}\right) - d(x,\{0\} \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \subseteq \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{\left[\sqrt{h_{r,s}}\right]}{h_{r,s}^{\alpha}} \geq \delta \right\}.$$

Since the set on the right hand side is finite and belongs to $I_2$, it follows that for $\frac{1}{2} < \alpha \leq 1, A_{j,k} \to \{0\} \left(S_{\theta_{r,s}}^{\alpha}(I_{2w})\right)$.

On the other hand, for $\frac{1}{2} < \alpha \leq 1$ and $x > 0$,

$$\frac{1}{h_{r,s}^{\alpha}} \sum_{(j,k)\in I_{r,s}} \sum \left| d\left(x, A_{j,k}\right) - d(x,\{0\}) \right| = \frac{(x^2 - 2x)\left[\sqrt{h_{r,s}}\right]}{h_{r,s}^{\alpha}} \to 0$$

And for $0 < \alpha < \frac{1}{2}, \frac{(x^2-2x)\left[\sqrt{h_{r,s}}\right]}{h_{r,s}^{\alpha}} \to \infty$.

Hence we have

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}^{\alpha}} \sum_{(j,k)\in I_{r,s}} \sum \left| d\left(x, A_{j,k}\right) - d(x,\{0\}) \right| \geq 0 \right\} = \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{(x^2 - 2x)\left[\sqrt{h_{r,s}}\right]}{h_{r,s}^{\alpha}} \geq 0 \right\}$$
$$= \{b, b+1, b+2, \dots\}$$

For some $b \in \mathbb{N}$ which belongs to $F(I)$, since $I_2$ is admissible. So $A_{j,k} \nrightarrow \{0\} \left(N_{\theta_{r,s}}^{\alpha}(I_{2w})\right)$.

(ii) Suppose that $\{A_{j,k}\} \in L_{\infty}^2$ and $A_{j,k} \to A \left(S_{\theta_{r,s}}^{\alpha}(I_{2w})\right)$. Then we can assume that

$$\left| d\left(x, A_{j,k}\right) - d(x,A) \right| \leq G$$

for each $x \in X$ and all $j,k \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\frac{1}{h_{r,s}^{\alpha}} \sum_{(j,k)\in I_{r,s}} \sum \left| d\left(x, A_{j,k}\right) - d(x,A) \right|$$
$$= \frac{1}{h_{r,s}^{\alpha}} \sum_{\substack{(j,k)\in I_{r,s} \\ \left|d\left(x,A_{j,k}\right)-d(x,A)\right| \geq \varepsilon}} \sum \left| d\left(x, A_{j,k}\right) - d(x,A) \right| + \frac{1}{h_{r,s}^{\alpha}} \sum_{\substack{(j,k)\in I_{r,s} \\ \left|d\left(x,A_{j,k}\right)-d(x,A)\right| < \varepsilon}} \sum \left| d\left(x, A_{j,k}\right) \right.$$
$$\left. - d(x,A) \right| \leq \frac{G}{h_{r,s}^{\alpha}} \left| \left\{ (j,k) \in I_{r,s} : \left| d\left(x, A_{j,k}\right) - d(x,A) \right| \geq \varepsilon \right\} \right| + \varepsilon.$$

Hence we have

$\left\{(r,s) \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}} \sum_{(j,k)\in I_{r,s}} \sum \left|d\left(x,A_{j,k}\right)-d(x,A)\right| \geq G\delta+\varepsilon\right\} \subset \left\{(r,s)\in\mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\{(j,k)\in\right.\right.$ Ir,s:dx,Aj,k−d(x,A)≥ε≥δ∈I2.

Therefore $A_{j,k} \to A\left(N_{\theta_{r,s}}^{\alpha}(I_{2w})\right)$.

(iii) Follows from (i) and (ii).

**Theorem 2.2:** Let $\theta_{r,s}$ be a double lacunary sequence and $\alpha$ be a fixed number such that $0<\alpha\leq 1$. If $lim\,\inf_{r,s} q_{r,s} > 1$, then $St_2^{\alpha}(I_{2w}) \subset S_{\theta_{r,s}}^{\alpha}(I_{2w})$.

**Proof:** Suppose first that $lim\,\inf_{r,s} q_{r,s} > 1$, then there exists a $\rho>0$ such that $q_{r,s}\geq 1+\rho$ for sufficiently large $r,s$, which implies that

$$\frac{h_{r,s}}{j_r k_s} \geq \frac{\rho}{1+\rho} \Rightarrow \left(\frac{h_{r,s}}{j_r k_s}\right)^{\alpha} \geq \left(\frac{\rho}{1+\rho}\right)^{\alpha} \Rightarrow \frac{1}{j_r k_s} \geq \frac{\rho^{\alpha}}{(1+\rho)^{\alpha}}\frac{1}{h_{r,s}^{\alpha}}.$$

$A_{j,k} \to A(St_2^{\alpha}(I_{2w}))$, then for every $\varepsilon>0$, for each $x\in X$, and for sufficiently large $r,s$, we have

$$\frac{1}{(j_r k_s)^{\alpha}}\left|\left\{j\leq j_r, k\leq k_s: \left|d\left(x,A_{j,k}\right)-d(x,A)\right|\geq\varepsilon\right\}\right| \geq \frac{1}{(j_r k_s)^{\alpha}}\left|\left\{(j,k)\in I_{r,s}: \left|d\left(x,A_{j,k}\right)-d(x,A)\right|\geq\varepsilon\right\}\right|$$
$$\geq \frac{\rho^{\alpha}}{(1+\rho)^{\alpha}}\frac{1}{h_{r,s}^{\alpha}}\left|\left\{(j,k)\in I_{r,s}: \left|d(x,A_{j,k})-d(x,A)\right|\geq\varepsilon\right\}\right|.$$

For $\delta>0$, we have

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\left\{(j,k)\in I_{r,s}: \left|d\left(x,A_{j,k}\right)-d(x,A)\right|\geq\varepsilon\right\}\right|\geq\delta\right\}$$
$$\subseteq \left\{(r,s)\in\mathbb{N}\times\mathbb{N}: \frac{1}{h_{r,s}^{\alpha}}\left|\left\{(j,k)\in I_{r,s}: \left|d\left(x,A_{j,k}\right)-d(x,A)\right|\geq\varepsilon\right\}\right|\geq\delta\frac{\rho^{\alpha}}{(1+\rho)^{\rho}}\right\}\in I_2.$$

This completes the proof.

**Theorem 2.3:** Let$\theta_{r,s}$ be a double lacunary sequence and the parameters $\alpha\ and\ \beta$ be fixed real numbers such that $0<\alpha\leq\beta\leq 1, then\ N_{\theta_{r,s}}^{\alpha}(I_{2w}) \subseteq N_{\theta_{r,s}}^{\beta}(I_{2w})$ and the inclusion is strict.

**Proof:** The inclusion part of the proof is easy. To show that the inclusion is strict define $\{A_{j,k}\}$ such that for $(\mathbb{R},d)$, $x>1$ and $A=\{0\}$,

$$A_{jk} = \begin{cases}(3x+5), j_{r-1}k_{s-1} < jk < j_{r-1}k_{s-1}+\sqrt{h_{r,s}} \\ \{0\}, \qquad otherwise\end{cases}.$$

Then $\{A_{j,k}\}\in N_{\theta_{r,s}}^{\beta}(I_{2w})$ for $\frac{1}{2}<\beta\leq 1$ but $\{A_{j,k}\}\notin N_{\theta_{r,s}}^{\alpha}(I_{2w})$ for $0<\alpha\leq\frac{1}{2}$.

**Theorem 2.4:** Let the parameters $\alpha$ and $\beta$ be fixed real numbers such that $0<\alpha\leq\beta\leq 1, St_2^{\beta}(I_{2w}) \subseteq N^{\alpha}(I_{2w})$.

**Proof:** For any double sequence $\{A_{j,k}\}$ and $\varepsilon>0$, we have

$$\frac{1}{(mn)^{\alpha}}\sum_{j=1}^{m}\sum_{k=1}^{n}\left|d\left(x,A_{j,k}\right)-d(x,A)\right| \geq \frac{1}{(mn)^{\alpha}}\left|\left\{j\leq m, k\leq n: \left|d\left(x,A_{j,k}\right)-d(x,A)\right|\geq\varepsilon\right\}\right|\varepsilon$$
$$\geq \frac{1}{(mn)^{\beta}}\left|\left\{j\leq m, k\leq n: \left|d\left(x,A_{j,k}\right)-d(x,A)\right|\geq\varepsilon\right\}\right|\varepsilon$$

and so

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}: \frac{1}{(mn)^{\alpha}}\sum_{j=1}^{m}\sum_{k=1}^{n}\left|d\left(x,A_{j,k}\right)-d(x,A)\right|\geq\delta\right\}$$
$$\subseteq \left\{(m,n)\in\mathbb{N}\times\mathbb{N}: \frac{1}{(mn)^{\beta}}\left|\left\{j\leq m, k\leq n: \left|d\left(x,A_{j,k}\right)-d(x,A)\right|\geq\varepsilon\right\}\right|\geq\frac{\delta}{\varepsilon}\right\}\in I_2.$$

This gives that$St_2^{\beta}(I_{2w}) \subseteq N^{\alpha}(I_{2w})$.

**Theorem 2.5:** Let $\theta_{r,s}$ be a double lacunary sequence and $\alpha$ be a fixed number such that $0<\alpha\leq 1$. If $\lim_{r,s\to\infty,\infty}\inf\frac{h_{r,s}^{\alpha}}{j_r k_s}>0$, then $St_2(I_{2w}) \subseteq S_{\theta_{r,s}}^{\alpha}(I_{2w})$.

**Proof:** Let $(X,d)$ be a metric space, $\theta_{r,s}$ be a double lacunary sequence and $A, A_{j,k}$ (for all $j,k\in\mathbb{N}$) be a non-empty closed subsets of $X$. If $\lim_{r,s\to\infty}\inf\frac{h_{r,s}^{\alpha}}{j_r k_s}>0$, then we can write

$$\{j \le j_r, k \le k_s : |d(x, A_{j,k}) - d(x, A)| \ge \varepsilon\} \supset \{(j,k) \in I_{r,s} : |d(x, A_{j,k}) - d(x, A)| \ge \varepsilon\}$$

$$\frac{1}{j_r k_s}\left|\{j \le j_r, k \le k_s : |d(x, A_{j,k}) - d(x, A)| \ge \varepsilon\}\right| \ge \frac{1}{j_r k_s}\left|\{(j,k) \in I_{r,s} : |d(x, A_{jk}) - d(x, A)| \ge \varepsilon\}\right|$$

$$= \frac{h_{r,s}^{\alpha}}{j_r k_s}\frac{1}{h_{r,s}^{\alpha}}\left|\{(j,k) \in I_{r,s} : |d(x, A_{j,k}) - d(x, A)| \ge \varepsilon\}\right|.$$

So

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}^{\alpha}}\left|\{(j,k) \in I_{r,s} : |d(x, A_{j,k}) - d(x, A)| \ge \varepsilon\}\right| \ge \delta\right\}$$
$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{j_r k_s}\left|\{j \le j_r, k \le k_s : |d(x, A_{j,k}) - d(x, A)| \ge \varepsilon\}\right| \ge \delta\frac{h_{r,s}^{\alpha}}{j_r k_s}\right\}$$

Which implies that $St_2(I_{2w}) \subseteq S_{\theta_{r,s}}^{\alpha}(I_{2w})$.

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