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Two-Step Implicit Hybrid Block Method for the Solution of $y'' = (x,\ y,\ y')$

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Abstract: *The solution for second order differential equation is studied using the two-step implicit hybrid block method. In this study, we adopted the interpolation of the approximation and the collocation of its differential equation. Using an orthogonal polynomial with respect to the weight function x^2 over an interval $[0, 1]$, this yields a linear multistep method with constant step-size. The developed methods are verified to be convergent, in addition, numerical examples are presented to demonstrate the accuracy and efficacy of the linear multi step method.*

Keywords: *Orthogonal Polynomials, Collocation, Interpolation, Linear Multistep Methods(LMMs).*

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I. Introduction

We consider the general second order Ordinary Differential Equations *(ODEs)* of the form:

$$y''(x) = f(x, y, y')(1)$$

Numerous problems such as (1) may not easily be solved analytically, consequently, numerical schemes are developed and used to find solutions to these problems. These equations are usually reduced to a system of first order ODE and are solved using numerical methods for first order ODEs.Certain studiessuch as [2-7], have established the direct solution of (1) without having to reduce (1) to a system of first order ODE and they proposed methods with different polynomials as their basis function. This study extendsthe work of yakusak et al (2015)in [1] to two-step implicitHybrid Block Method(HBM) via Multistep Collocation technique.

In succeeding sections, we derive the linear multi-step method using orthogonal polynomials with respect to the weight function x^2 over a specified interval. The derived method is analyzed, numerical examples are considered and used to validate the method and a resulting conclusion is established.

II. Methodology

In this study, we consider an Orthogonal Polynomial over the interval [0, 1], with respect to the weight function x^2 of a single variable as our approximate solution, this is written in the form

$$y(x) = \sum_{j=0}^{s+r-1} a_j Q(x)^j (2)$$

$$y''(x) = \sum_{j=0}^{s+r-1} j(j-1) a_j Q(x)^{j-2} (3)$$

Putting (4) into (1) yield

$$f(x, y, y') = \sum_{j=0}^{s+r-1} j(j-1) a_j Q(x)^{j-2} (4)$$

The solution to (1) issolved on the partition:

$$\pi_N: a = x_0 < x_1 < x_2 \ \ldots . < x_n < x_{n+1} \ldots .. < x_{n+1} = b$$

Witha constant step size given as

$$h = x_{n+i} - x_n, \quad n = 0, 1, 2, \ldots N$$

Interpolating (2) at $x_{n+s}, s = 1, \frac{3}{2}$ and collocating (4) at $x_{n+r}, r = 0, 1, \frac{3}{2}, 2$ gives

$$\sum_{j=0}^{s+r-1} \alpha_j x^j = y_{n+r} \ (5)$$

$$\sum_{j=0}^{s+r-1} j\,(j-1)\alpha_j\, Q(x)^{j-2} = f_{n+r} \quad (6)$$

Where *s* is the number of interpolation and *r* is the number of collocation points.
Solving (5) and (6) for $\alpha_{j,s}$ and substituting back in (2) we obtain the continuous *LMM* of the form

$$y(x) = \sum_{j=0}^{1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{2} \beta_j(x) f_{n+j} \quad (7)$$

Where the coefficient y_{n+j}, and f_{n+j} are given as

$$\alpha_1(t) = 3 - 2t\alpha_{\frac{3}{2}}(t) = -2 + 2t$$

$$\beta_0(t) = \frac{1}{12} - \frac{119}{360}t + \frac{1}{2}t^2 - \frac{13}{36}t^3 + \frac{1}{8}t^4 - \frac{1}{60}t^5$$

$$\beta_1(t) = \frac{13}{16} - \frac{319}{240}t + t^3 - \frac{7}{12}t^4 + \frac{1}{10}t^5$$

$$\beta_{\frac{3}{2}}(t) = -\frac{5}{24} + \frac{203}{360}t - \frac{8}{9}t^3 + \frac{2}{3}t^4 - \frac{2}{15}t^5$$

$$\beta_2(t) = \frac{1}{16} - \frac{37}{240}t + \frac{1}{4}t^4 + \frac{1}{20}t^5$$

Where $t = \frac{x - x_n}{h}$

Solving for the independent solution $y(x)$ in (7) gives a continuous block method. Where the coefficient of f_{n+j} is given by

$$\sigma_0 = \frac{1}{2}t^2 - \frac{13}{36}t^3 + \frac{1}{8}t^4 - \frac{1}{60}t^5$$

$$\sigma_1(t) = t^3 - \frac{7}{12}t^4 + \frac{1}{10}t^5$$

$$\sigma_{\frac{3}{2}}(t) = -\frac{8}{9}t^3 + \frac{2}{3}t^4 - \frac{2}{15}t^5$$

$$\sigma_2(t) = \frac{1}{4}t^4 + \frac{1}{20}t^5$$

Evaluating the continuous block method at $t = 1, \frac{3}{2}, 2$ gives the discrete block method as

$$y_{n+1} = y_n + y'_n + h^2\left(\frac{89}{360}f_n + \frac{31}{60}f_{n+1} - \frac{16}{45}f_{n+\frac{3}{2}} + \frac{11}{120}f_{n+2}\right)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}y'_n + h^2\left(\frac{33}{80}f_n + \frac{189}{160}f_{n+1} - \frac{51}{80}f_{n+\frac{3}{2}} + \frac{27}{160}f_{n+2}\right)$$

$$y_{n+2} = y_n + 2y'_n + h^2\left(\frac{26}{45}f_n + \frac{28}{15}f_{n+1} - \frac{32}{45}f_{n+\frac{3}{2}} + \frac{4}{15}f_{n+2}\right)$$

$$y'_{n+1} = y'_n + h\left(\frac{1}{3}f_n + \frac{7}{6}f_{n+1} - \frac{2}{3}f_{n+\frac{3}{2}} + \frac{1}{6}f_{n+2}\right) \quad (8)$$

$$y'_{n+\frac{3}{2}} = y'_n + h\left(\frac{21}{64}f_n + \frac{45}{32}f_{n+1} - \frac{3}{8}f_{n+\frac{3}{2}} + \frac{9}{64}f_{n+2}\right)$$

$$y'_{n+2} = y'_n + h\left(\frac{1}{2}f_n + \frac{4}{3}f_{n+1} + \frac{1}{3}f_{n+2}\right)$$

III. Analysis of the Method

Order of the Method

Let the linear operator $L\{y(x){:}\,h\}$ associated with method (8) expanding in Taylor series and comparing the coefficient of h gives

$$L\{y(x){:}\,h\} = c_0 y^0(x) + c_1 h^1 y^1(x) + \cdots + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + c_{p+2} h^{p+2} y^{p+2}(x)$$

Definition 1

The linear operator *L* and the associated *LMMs*are said to be of order P if
$c_0 = c_1 = c_2 = \cdots c_p = c_{p+1} = 0$ and$c_{p+2} \neq 0$
$c_{p+1}$is called the error constant and Local truncation error is given by

$$t_{n+k} = c_{p+2} h^{(P+2)} y^{(P+2)}(x_n) + 0(h^{p+3})$$

Following the above definition our method is of order 4

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0$$

and the error constant is given as

$$c_6 = (-\frac{1}{160}, -\frac{117}{10240}, -\frac{1}{60}, -\frac{31}{2880}, -\frac{51}{5120}, -\frac{1}{90})$$

Zero – Stability

Definition 2

The block method (8) is said to be zero – stable if the characteristic polynomial

$$\rho(z) = det\left(zA^{(0)} - E\right)$$

Satisfies$/z_s/ \leq 1$, gave multiplicity not exceeding the order of differential equation as $h \to 0$

Following the above definition our method (8) is zero stable.

Convergence

The convergence of the continuous hybrid block method is considered in light of the basic properties discussed above in conjunction with the fundamental theorem of Dahlquist[8] for *LMM*; we state the Dahlquist theorem without proof.

Theorem 3.1 The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.

Following the theorem 3.1 above shows that both methods are convergent

V.Numerical examples

We implement our method on second order ordinary differential equations of the form.

Example 1

$$y'' - y' = 0,$$
$$y(0) = 0, y'(0) = -1, \quad h = 0.1$$

Exact Solution $y(x) = 1 - e^x$

Example 2

$$y'' + 1001y' + 100y = 0,$$
$$y(x) = 1, y'(x) = -1, h = 0.55$$

Exact Solution $y(x) = e^{-x}$

Table 1: Numerical Solution for Example 1

X	Exact	Our Method
0.1	-0.105170918	-0.105170925
0.2	-0.221402758	-0.221402778
0.3	-0.349858807	-0.349858852
0.4	-0.491824697	-0.491824775
0.5	-0.64872127	-0.648721396
0.6	-0.822118800	-0.822118986
0.7	-1.013752707	-1.013752977
0.8	-1.225540928	-1.225541294
0.9	-1.459603111	-1.459603604
1.0	-1.718281228	-1.718282473

Table 2: Comparison of Error for Example 1

X	Our Method	[3]
0.1	7.30 E -09	1.61 E -08
0.2	2.04 E -08	3.51 E -08
0.3	4.39 E -08	2.37 E -07
0.4	7.67 E -08	2.64 E -07
0.5	1.24 E -07	2.96 E -07
0.6	1.86 E -07	3.34 E -07
0.7	2.67 E -07	3.78 E -07
0.8	3.66 E -07	4.30 E -07
0.9	4.93 E-07	4.91 E -07
1.0	6.45 E -07	5.61 E -07

Table 1: Numerical Solution for Example 2

X	Exact	Our Method
0.1	0.9048374187	0.9048374178
0.2	0.8187307531	0.8187307527
0.3	0.7081822070	0.7081822010
0.4	0.6703200560	0.6703200454
0.5	0.6065306597	0.6065306590
0.6	0.5488116361	0.5488116353
0.7	0.4965853038	0.4965853030
0.8	0.4493289641	0.4493289633
0.9	0.4065696597	0.4065696589
1.0	0.3678794412	0.3678794403

Table 2: Comparison of Error for Example 2

X	Our Method	[3]
0.1	2.10 E -10	2.35 E -10
0.2	4.10 E -10	4.77 E -10
0.3	6.10 E -10	5.81 E -10
0.4	6.10 E -10	7.35 E -10
0.5	7.10 E -10	8.26 E -10
0.6	8.10 E -10	8.95 E -10
0.7	8.10 E -10	9.14 E -10
0.8	8.10 E -10	1.01 E -09
0.9	8.10 E -10	1.04 E -09
1.0	9.10 E -10	1.07 E-09

IV. Conclusion

The implementation of the above scheme is done with the aid of maple software. As we can see in the above results, the numerical solution behaves like the theoretical solution. We implement the scheme on two numerical examples and the method is found to be convergent. Our method is therefore favorable.

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