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Numerical Solutions of Some Second Order Partial Differential Equations using Non-Standard Exponent Values of the Generalized Multiquadric Radial Basis Function

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Abstract: *The generalized multiquadrics radial basis function (GMQ-RBF) methods are numerical methods which are used independently or combined with other numerical methods to develop hybrid numerical methods for approximating partial differential equations (PDEs), integral equations (IQs), integro-differential equations (IDEs) and interpolation problems. The standard GMQ-RBFs are well known and are commonly applied for approximating the solutions of some mathematical problems, however, GMQ-RBFs having non-standard exponents appear in literature but are not commonly used. In this paper, two GMQ-RBFs with non-standard exponentsare used for the space discretization of some time-dependent PDEs and combined with the fourth order Runge-Kutta method which is used as a time-stepping method to propose two radial basis function method of lines (RBF-MOLs).Theproposed methods are implemented in MATLAB and applied to approximate the solution of some time-dependent PDEs in one space dimension. Our proposed methods compared favourably with some numerical results obtained from some standard generalized RBF-MOLs.*

Keywords: *Radial Basis Functions, Generalized Multiquadric Radial Basis Functions, Radial Basis Function Method of Lines.*

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I. Introduction

A radial basis function (RBF) is a real valued function $\phi: \mathbb{R}^d \longrightarrow \mathbb{R}$, whose values depends only on distance $x \in \mathbb{R}^d$ and some fixed point $x_j \in \mathbb{R}^d, j = 1,2, \ldots, N$ (centres) such that

$$\phi(x, x_j) = \phi(\|x - x_j\|) = \phi(r) \tag{1.1}$$

where r is generally the Euclidean distance. Radial basis function (RBF) methods were derived for the purpose of multivariate scattered interpolation, but in recent times they are applied in different areas of mathematical sciences and engineering such as PDEs etc.[1]. Hardy[2], introduced the multiquadric (MQ) RBF method, his work was primarily concerned with application of scattered data interpolation in geodesy and mapping[3]. Other RBFs such as the thin plate splines (TPS) and the surface splines were later introduced by[4, 5]. Franke[6], compared different interpolation methods and rated MQ and TPS as the best in terms of the ease of implementation, storage accuracy and visual pleasantness of the surface. This discovery paved way for the application of RBFs in different scientific computing communities, especially the MQ RBF.

Kansa,[7,8]was the first to develop an RBF collocation scheme for approximating the solutions of the elliptic, parabolic and hyperbolic PDEs using the MQ RBF. Kansa's breakthrough lead to the application of RBFs for approximating the solutions of integral equations, integro-differential equations etc. in different mathematical and engineering disciplines[9].

A generalized version of the MQ RBF (GMQ) is given by

$$\phi(r, \varepsilon) = (1 + \varepsilon^2 r^2)^\beta \tag{1.2}$$

where the exponent β may be any real number except non-negative integers[10]. Equation (1.2) is referred to as MQ RBF, inverse multiquadric (IMQ), inverse quadratic (IQ), generalized inverse multiquadric (GIMQ) if $\beta = \frac{1}{2}, -\frac{1}{2}, -1 \text{ and } -2$ respectively.The GMQ-RBF is strictly positive definite if $\beta < 0$ and conditionally positive definite of order $\lceil \beta \rceil$ if $\beta > 0$ [3]. Positive definite RBFs do not require a polynomial term to be appended to its interpolation matrix to make it invertible, while the interpolation matrix of conditionally positive are appended with a polynomial term, however, many researchers have used conditionally positive definite RBFs without appending the polynomial term, yet good approximations were obtained[10].

The MQ RBF and the IMQ have been applied to solve many PDEs and integral equations see[10, 11, 12]etc., while the IQ and GIMQ-RBFs were utilize by[13, 14] to approximate some time-dependent PDEs. The GMQ-RBFs with the value of the exponent $\beta = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$ are in recent times explored by[9] in conjunction with the advanpix software for approximating PDEs without appending any polynomial term, however, little experiments have been carried out with GMQ-RBFs having non-standard exponent β. Although some researchers such as[15, 16, 17]have reported good results with some non-standard values of GMQ-RBFs exponents for $\beta = 1.03$ and $\beta = 1.99$ respectively, much applications are not found in literature about them especially for approximating PDEs.

In this paper, we wish to apply the values of the exponent $\beta = 1.03$ and $\beta = 1.99$ in equation (1.2) to approximate the solutions of some time-dependent PDEs via the method of lines (MOLs) in one space dimension. We shall obtain our test problems from[10] for the purpose of comparison.

The rest of the paper is organized as follows, the derivation of the methods is considered in Section 2, while the results are presented in Section 3 and finally discussion and conclusion are done in Sections 4 and5respectively.

II. Methods

The derivation of the differentiation matrix of the GMQ RBFs having non-standard values of the exponent $\beta = 1.03$ and $\beta = 1.99$ for discretizing the space derivatives ofsometime-dependent PDEs in one space dimension are provided in this section.

To solve a PDE with an RBF method, the space derivatives are discretized using a differentiation matrix which depends on both the evaluation and interpolation matrices of the required RBF. First we make an assumption that if $u: \mathbb{R}^d \longrightarrow \mathbb{R}$ is the unknown solution, it can be approximated with an RBF interpolant $s: \mathbb{R}^d \longrightarrow \mathbb{R}$defined by

$$s(x) = u(x) \tag{2.1}$$

where

$$s(x) = \sum_{j=1}^{N} \lambda_j \phi\left(\|x - x_j\|\right) + p(x), \tag{2.2}$$

$\|\cdot\|$ denotes the Euclidean norm, $p(x) \in \mathcal{P}_m^d$ is a polynomial of degree $m-1$ and $\lambda_j, j = 1,2,3,\ldots,N$ is a vector of an unknown to be found. However, we wish to use equation (2.2) without appending the polynomial term $p(x)$ as explained in Sarra and Kansa (2009). Thus, equation (2.2) can be written as

$$s(x) = \sum_{j=1}^{N} \lambda_j \phi\left(\|x - x_j\|\right). \tag{2.3}$$

Substituting equation (2.3) in equation (2.1) and expanding for each $x = x_i, i = 1,2,3,\ldots,N$, and $j = 1,2,3,\ldots,N$ gives the system of linear equations in matrix form

$$\begin{bmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_1 - x_2\|) & \cdots & \phi(\|x_1 - x_N\|) \\ \phi(\|x_2 - x_1\|) & \phi(\|x_2 - x_2\|) & \cdots & \phi(\|x_2 - x_N\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|x_N - x_1\|) & \phi(\|x_N - x_2\|) & \cdots & \phi(\|x_N - x_N\|) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \tag{2.4}$$

Equation (2.4) is the general interpolation matrix of an RBF. It can be expressed in vector-matrix form as

$$A\lambda = u. \tag{2.5}$$

The unknown vector $\lambda$is obtained from equation (2.5) as shown in equation (2.6)

$$\lambda = A^{-1}u \tag{2.6}$$

The evaluation matrix is obtained by evaluating equation (2.3) for each data point$x_i, i = 1,2,3,\ldots,M$, $j = 1,2,3,\ldots,N$. However, to ensure a symmetric evaluation matrix, N data points and N evaluation points are used. The evaluation matrix is expressed as

$$\sum_{j=1}^{N} \lambda_j \phi\left(\|x_i - x_j\|\right) = H\lambda_j \tag{2.7}$$

where H has the entries $h_{ij} = \phi\left(\|x_i - x_j\|\right)$.

Differentiating (2.7), we get the differentiation matrix

$$\sum_{j=1}^{N} \lambda_j \frac{\partial}{\partial x_i} \phi\left(\|x_i - x_j\|\right) = \frac{\partial}{\partial x_i} H\lambda_j \tag{2.8}$$

Equation (2.8) can be differentiated the number of times requiredto get the order of the derivative of interest. Thus, relating (2.8) and (2.1) shows that

$$\frac{\partial}{\partial x_i}u(x_i) \approx \frac{\partial}{\partial x_i}s(x_i) = \frac{\partial}{\partial x_i}H\lambda_j. \tag{2.9}$$

Substituting (2.6) in (2.9) yields

$$\frac{\partial}{\partial x_i}u = \frac{\partial}{\partial x_i}HA^{-1}u.$$

Let

$$D = \frac{\partial}{\partial x_i}HA^{-1} \tag{2.10}$$

Equation (2.10) is called the differentiation matrix, it is used for approximating the derivatives of a given PDE. Substituting

$$\phi(r) = \left(1 + \left(\varepsilon\|x_i - x_j\|\right)^2\right)^{1.03} \tag{2.11}$$

and

$$\phi(r) = \left(1 + \left(\varepsilon\|x_i - x_j\|\right)^2\right)^{1.99} \tag{2.12}$$

$i = 1,2,3, \dots, N$ and $j = 1,2,3, \dots, N$,

We get the following basis functions which can be substituted in equation (2.4) and (2.7) to get the desired interpolation and evaluation matrices respectively which are used for the formulation of the differentiation matrices as shown in equation (2.10). In one dimension, the interpolation matrices for the non-standard GMQ-RBFs are given below

$$\begin{bmatrix} (1 + (\varepsilon|x_1 - x_1|)^2)^n & (1 + (\varepsilon|x_1 - x_2|)^2)^n & \cdots & (1 + (\varepsilon|x_1 - x_N|)^2)^n \\ (1 + (\varepsilon|x_2 - x_1|)^2)^n & (1 + (\varepsilon|x_2 - x_2|)^2)^n & \cdots & (1 + (\varepsilon|x_2 - x_N|)^2)^n \\ \vdots & \vdots & \ddots & \vdots \\ (1 + (\varepsilon|x_N - x_1|)^2)^n & (1 + (\varepsilon|x_N - x_2|)^2)^n & \cdots & (1 + (\varepsilon|x_N - x_N|)^2)^n \end{bmatrix} \tag{2.13}$$

where
$n = 1.03$ and 1.99

Existence and Uniquenessof Interpolation Matrix of RBFs: Forany RBF applied to discretize the space derivatives of a PDE to exist and be unique, the interpolation matrix has to be invertible[10]. There are many methods for characterizing the existence and uniqueness of an interpolation matrix[3], however, we shall verify that the basic functions of the RBFs of interest are completely monotone.

Theorem 1: Completely Monotone Functions[10]
A function $\phi(r)$ is completely monotone on $[0, \infty)$ if
(i) $\phi \in C[0, \infty)$,
(ii) $\phi \in C^\infty(0, \infty)$,
(iii) $(-1)^\ell \phi^{(\ell)}(r) \geq 0$,
where $r > 0$ and $\ell = 0,1,2, \dots$.
The RBFs we wish to use are conditionally positive definite, so we shall verify that their basic functions are completely monotone using Theorem 2 below.

Theorem 2[3]
Suppose $\phi(r)$is completely monotone, then
$\phi(r) = (-1)^{\lceil\beta\rceil}(1 + r)^\beta, \qquad 0 < \beta \notin \mathbb{N}$,
imply
$\phi^{(\ell)}(r) = (-1)^{\lceil\beta\rceil}\beta(\beta - 1) \dots (\beta - \ell + 1)(1 + r)^{\beta - \ell}$
so that

$$(-1)^{\lceil\beta\rceil}\phi^{(\lceil\beta\rceil)}(r) = \beta(\beta - 1) \dots (\beta - \lceil\beta\rceil + 1)(1 + r)^{\beta - \lceil\beta\rceil} \tag{2.14}$$

where $\lceil\beta\rceil$ means the least integer greater than β.
If $\beta = 1.03$ and $\beta = 1.99$, then both $\lceil 1.03 \rceil$ and $\lceil 1.99 \rceil$ are equal to 2.
Thus equation (2.14) in both cases reduces to

$$(-1)^2\phi^{(2)}(r) = (1.03)(0.03)\frac{1}{(1 + r)^{0.97}} \geq 0 \tag{2.15}$$

and

$$(-1)^2\phi^{(2)}(r) = (1.99)(0.99)\frac{1}{(1 + r)^{0.01}} \geq 0 \tag{2.16}$$

Equation (2.15) and (2.16) shows that Theorem 1 is verified.

Discretizing Derivatives with RBFs: To discretize a derivative using RBFs $\phi[r(x)]$, the chain rule for the first two derivatives according to Sarra and Kansa (2009) are given as

$$\frac{\partial \phi}{\partial x_i} = \frac{d\phi}{dr}\frac{\partial r}{\partial x_i} \tag{2.17}$$

and

$$\frac{\partial^2 \phi}{\partial {x_i}^2} = \frac{d\phi}{dr}\frac{\partial^2 r}{\partial {x_i}^2} + \frac{d^2\phi}{\partial r^2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 \tag{2.18}$$

where

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

and

$$\frac{\partial^2 r}{\partial {x_i}^2} = \frac{1-\left(\frac{\partial r}{\partial x_i}\right)^2}{r}.$$

For the GMQ RBF having the value of $\beta = 1.03$,

$$\frac{d\phi}{dr} = 2.06\varepsilon^2 r(1+\varepsilon^2 r^2)^{0.03}$$

and

$$\frac{d^2\phi}{\partial r^2} = 2.06\varepsilon^2 r(1+\varepsilon^2 r^2)^{0.03} + \frac{0.1236\varepsilon^4 r^2}{(1+\varepsilon^2 r^2)^{0.97}}$$

Similarly, for the GMQ RBF having the value of $\beta = 1.99$

$$\frac{d\phi}{dr} = 3.98\varepsilon^2 r(1+\varepsilon^2 r^2)^{0.99}$$

and

$$\frac{d^2\phi}{dr^2} = 3.98\varepsilon^2 r(1+\varepsilon^2 r^2)^{0.99} + \frac{7.8804\varepsilon^4 r^2}{(1+\varepsilon^2 r^2)^{0.01}}$$

Algorithm for Approximating the Solution of Time-Dependent PDEs

To solve time-dependent PDEs considered in this paper using the method of lines (MOLs), once the space discretization is performed using RBFs as described above, the PDE can be written as

$$\frac{\partial u}{\partial t} + \{\mathcal{L}u + \mathcal{B}u\} = 0 \tag{2.19}$$

where

$$\mathcal{L}u = \sum_{j=1}^{N_I} \lambda_j\,(t)\mathcal{L}(\phi|x_i - x_j|), \qquad i = 1, 2, \ldots, N_I \tag{2.20}$$

and

$$\mathcal{B}u = \sum_{j=N_{I+1}}^{N} \lambda_j\,(t)\mathcal{B}(\phi|x_i - x_j|), \qquad i = N_{I+1}, \ldots, N, \tag{2.21}$$

$\mathcal{L}$ and $\mathcal{B}$ are the differential operators applied to the interior and boundary points respectively.

Equation (2.20) – (2.22) can be written as a single equation as

$$\frac{du}{dt} = \frac{\partial}{\partial x_i} H\lambda_j \tag{2.22}$$

where

$$\frac{\partial}{\partial x_i} H = \{\mathcal{L}u + \mathcal{B}u\}.$$

Equation (2.4) can be recast in this case as

$$A\lambda = u, \qquad [u_1(t), u_1(t), \ldots, u_N(t)], \tag{2.23}$$

equation (2.23) can be written as equation (2.6) and substituting in (2.22) leads to

$$\frac{du}{dt} = \frac{\partial}{\partial x_i} HA^{-1}u \tag{2.24}$$

or

$$\frac{du}{dt} = Du \tag{2.25}$$

Equation (2.25) leads to a system of ordinary differential equations (ODEs) which can be integrated using a suitable time-stepping method. In this paper,the fourth order Runge-Kutta method is used to integrate the resulting system of ODEs.

III. Results

In this section, three test problems on some time-dependent PDEs in one space dimension are implemented in MATLAB 2017b on WINDOWS 8 operating system and displayed in Graphs and Tables. The test problems and parameters values are obtained from[10, 18].These results are compared with the MQ-RBF-MOLs from[10] and the IQ and GIMQ RBF-MOLs from[12].

The following symbols are found in all the Tables

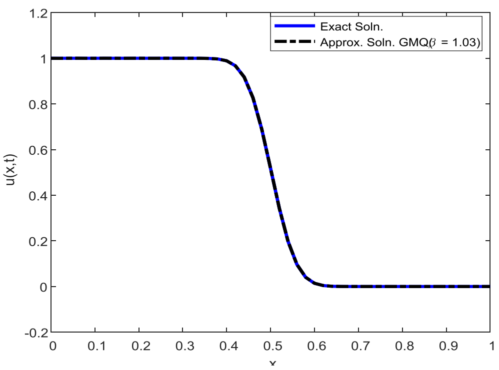
S/No.	Symbol	Meaning
1	Δt	Change in Time
2	FT	Final Time
3	N	Number of Data Points
4	ε	Shape Parameter
5	MPE	Maximum Point-wise Error

Example 1: The Linear Advection-Diffusion Equation[10]

The linear advection-diffusion equation is given by

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}, \quad v > 0 \tag{3.1}$$

where $a = 1$ and $v = 0.002$,
approximated on the domain
$\Omega = [0,1]$
the initial condition is
$u(x, 0) = 0$
the boundary conditions are
$u(0, t) = 1$
and
$u(x, t) = \frac{1}{2}\left[erfc\left(\frac{1-t}{2\sqrt{vt}}\right) + exp\left(\frac{1}{v}\right) erfc\left(\frac{1+t}{2\sqrt{vt}}\right)\right].$
The exact solution is given by
$u(x, t) = \frac{1}{2}\left[\mathrm{erfc}\left(\frac{x-t}{2\sqrt{vt}}\right) + \exp\left(\frac{1}{v}\right) \mathrm{erfc}\left(\frac{x+t}{2\sqrt{vt}}\right)\right].$
The results for Example1 are given in Figures 1, 2 and Table 1



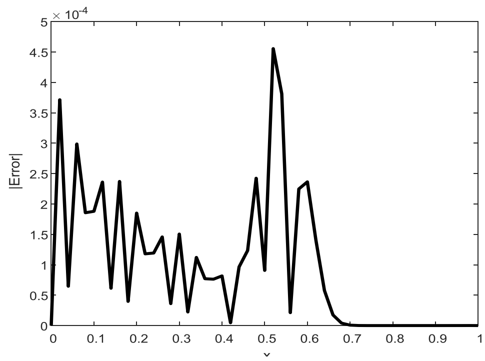
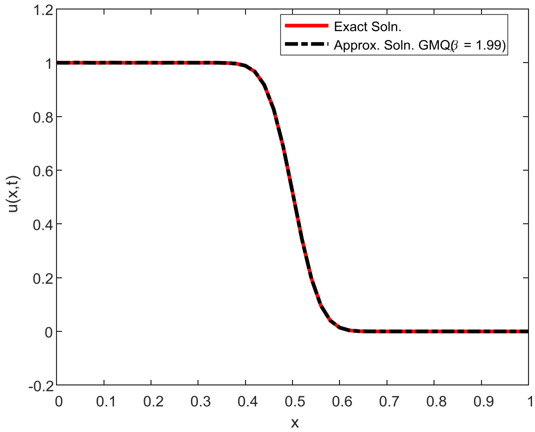


Fig. 1(a): Numerical solution usingGMQ RBF-MOLs with $\beta = 1.03$ versus the exact solution.

Fig. 1(b): Point-wise error of theGMQ RBF-MOLS with $\beta = 1.03$ for Example 4.1



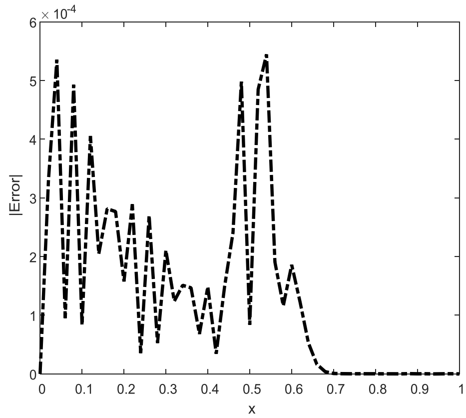


Fig. 2(a): Numerical solution usingGMQ RBF-MOLs with $\beta = 1.99$ versus the exact solution.
Fig. 2(b): Point-wise error of theGMQ RBF-MOLS with $\beta = 1.99$ for Example 4.1

Table 1: Comparison of MQ, IMQ, IQ, GIMQ and GMQ ($\beta = 1.03$ and $\beta = 1.99$) RBF-MOLS for Example 4.1

S/No	RBF-MOLs	N	Δt	FT	ε	MPE	Source
1	MQ	51	5.0×10^{-3}	5.0×10^{-1}	6.0	4.6960×10^{-4}	[10]
2	IMQ	51	5.0×10^{-3}	5.0×10^{-1}	5.1	3.5065×10^{-4}	[13]
3	IQ	51	5.0×10^{-3}	5.0×10^{-1}	5.0	4.2310×10^{-4}	[13]
4	GIMQ	51	5.0×10^{-3}	5.0×10^{-1}	4.5	4.5420×10^{-4}	[13]
5	GMQ with $\beta = 1.03$	51	5.0×10^{-3}	5.0×10^{-1}	6.0	4.5560×10^{-4}	
6	GMQ with $\beta = 1.99$	51	5.0×10^{-3}	5.0×10^{-1}	6.9	5.4430×10^{-4}	

Example 4.2: A Nonhomogenous Heat Equation[18]

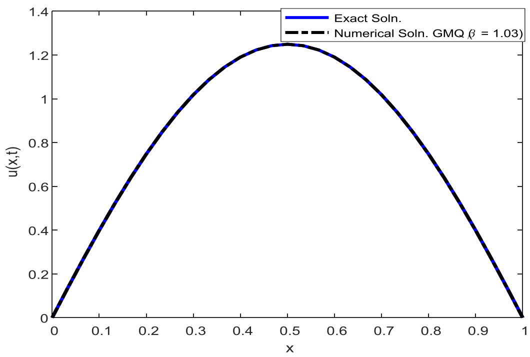
A problem involving a nonhomogenous heat equation is given below

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2, \tag{3.2}$$

on the domain
$\Omega = [0,1]$
with the initial condition

$$u(x,0) = \sin(\pi x) = x(1-x)$$

and boundary conditions
$u(0,t) = 0$, $u(1,t) = 0$.
The exact solution is given by
$u(x,t) = \exp(-\pi^2 t)\sin(\pi x) + x(1-x)$.
The results for Example 3.2 are given in the Figures 3, 4 and Table



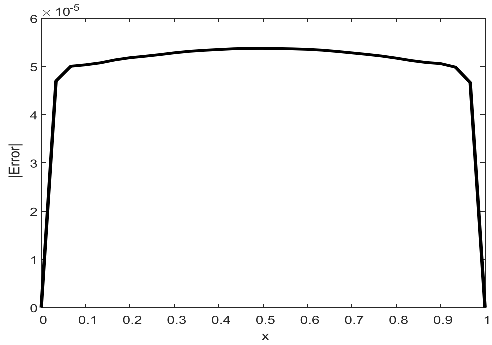
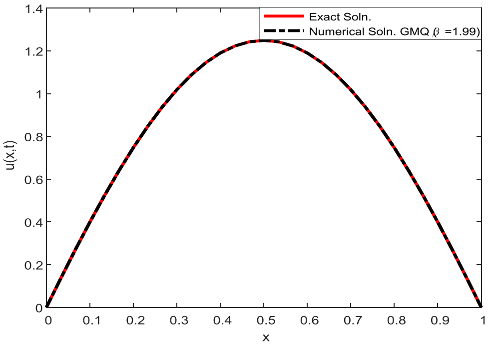


Fig. 3(a): Numerical solution usingGMQ RBF-MOLs with $\beta = 1.03$ versus the exact solution.

Fig. 3(b): Point-wise error of theGMQ RBF-MOLS with $\beta = 1.03$ for Example 4.2



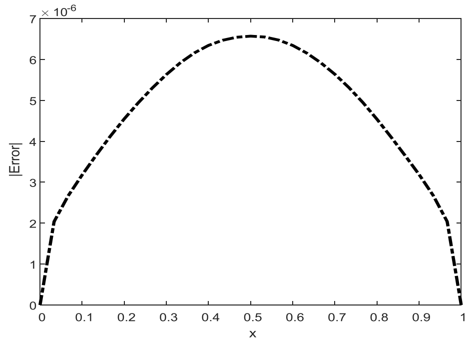


Fig. 4(a): Numerical solution usingGMQ RBF-MOLs with $\beta = 1.99$ versus the exact solution.
Fig. 4(b): Point-wise error of theGMQ RBF-MOLS with $\beta = 1.99$ for Example 4.2

Table 2: Comparison of MQ, IMQ, IQ, GIMQ and GMQ ($\beta = 1.03$ and $\beta = 1.99$) Example 4.2

S/No	RBF-MOLs	N	Δt	FT	ε	MPE	Source
1	MQ	31	5.0×10^{-7}	5.0×10^{-4}	5.0	2.5380×10^{-4}	10
2	IMQ	31	5.0×10^{-7}	5.0×10^{-4}	5.0	1.6630×10^{-4}	13
3	IQ	31	5.0×10^{-7}	5.0×10^{-4}	5.0	7.4550×10^{-5}	13
4	GIMQ	31	5.0×10^{-7}	5.0×10^{-4}	3.0	1.9780×10^{-5}	13
5	GMQ with $\beta = 1.03$	31	5.0×10^{-7}	5.0×10^{-4}	1.0	1.4500×10^{-5}	
6	GMQ with $\beta = 1.99$	31	5.0×10^{-7}	5.0×10^{-4}	1.2	9.5000×10^{-6}	

Example 4.3: The Burgers' Equation[10]

The Burger's equation is given by

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = v\frac{\partial^2 u}{\partial x^2}, \quad v > 0 \tag{3.3}$$

On the domain

$\Omega = [-1,1]$,

with the initial condition

$u(x,0) = 0$

and boundary conditions

$$u(-1,t) = g(t), u(1,t) = h(t)$$

The boundary conditions are taken from the exact solution

$$u(x,t) = \frac{0.1\exp(a) + 0.5\exp(b) + \exp(c)}{\exp(a) + \exp(b) + \exp(c)}$$

where

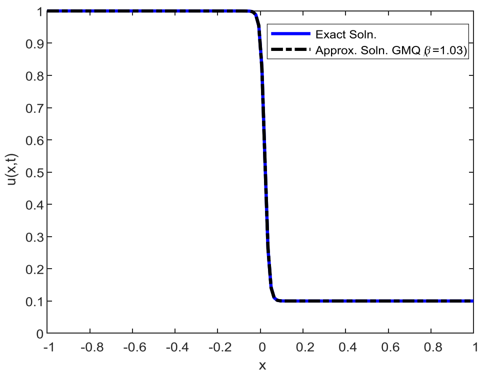
$$a = \frac{-(x + 0.5 + 4.95t)}{2v},$$

$$b = \frac{-(x + 0.5)}{4v},$$

$$c = \frac{-(x + 0.625 + 0.75t)}{2v},$$

$v = 4.4 \times 10^{-3}$

The results for Example 3.3 are provided in Figures 5, 6 and Table 3



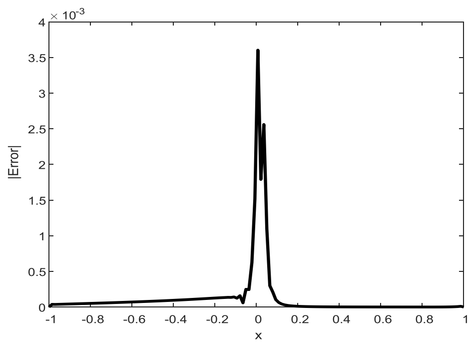
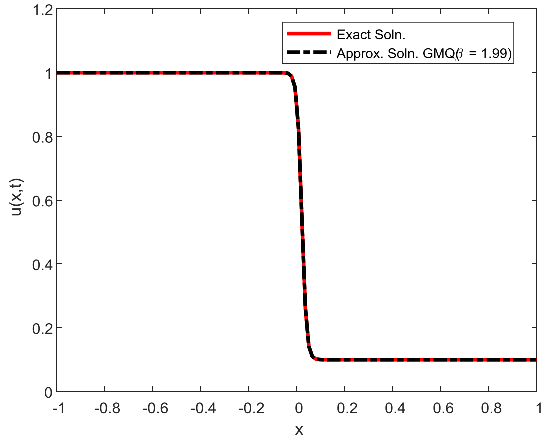


Fig. 5(a): Numerical solution usingGMQ RBF-MOLs with $\beta = 1.03$ versus the exact solution.
Fig. 5(b): Point-wise error of theGMQ RBF-MOLS with $\beta = 1.03$ for Example 4.3



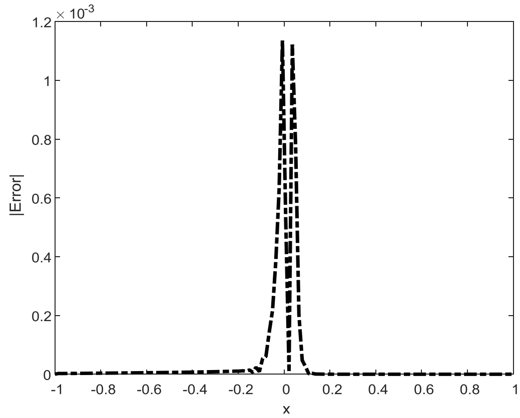


Fig. 6(a): Numerical solution usingGMQ RBF-MOLs with $\beta = 1.99$ versus the exact solution.
Fig. 6(b): Point-wise error of theGMQ RBF-MOLS with $\beta = 1.99$ for Example 4.2

Table 3: Comparison of MQ, IMQ, IQ, GIMQ and GMQ ($\beta = 1.03$ and $\beta = 1.99$) for Example 4.3

S/No	RBF-MOLs	N	Δt	FT	ε	MPE	Source
1	MQ	140	1.0×10^{-5}	1.2	11	6.6230×10^{-4}	[10]
2	IMQ	140	1.0×10^{-5}		8.0	7.5159×10^{-4}	[13]
3	IQ	140	1.0×10^{-5}		8.0	1.1000×10^{-3}	[13]
4	GIMQ	140	1.0×10^{-5}		7.0	1.3000×10^{-3}	[13]
5	GMQ with $\beta = 1.03$	140	1.0×10^{-5}		80	3.5700×10^{-3}	
6	GMQ with $\beta = 1.99$		1.0×10^{-5}		110	1.1500×10^{-3}	

IV. Discussion

Discussion

Example 4.1 is a linear advection-diffusion equation. To approximate this equation by RBF-MOLs, the domain was partitioned into $N = 51$ uniformly spaced centres and the spatial derivatives were approximated using two non-standard exponent valuesGMQ RBFs. Since the advection-diffusion equation is linear, the entire space derivatives were discretized using one differential operator and one differentiation matrix. The shape parameters $\varepsilon = 6.0$ and $\varepsilon = 6.9$ were used for the GMQ RBF-MOLs with $\beta = 1.03$ and $\beta = 1.99$ respectively to approximate the solution of the problem. In the course of implementing the program in MATLAB, it was observed that values of the shape parameter less than 5 tend not to yield good approximations for this problem. The brute force method was used to obtain a suitable estimate for the shape parameter for the proposed RBF methods. The fourth order Runge-Kutta method was used to integrate the system of ODEs that resulted from the spatial discretization of the PDE. We chose the temporal step size to be 5.0×10^{-3} and used it to advance the problem to a final time 5.0×10^{-1}. For the purpose of comparison, the plots of the exact and numerical solutions are provided in Figures 1 and 2 on the same graph. The maximum point-wise errors obtained by using the GMQ RBF-MOLs with $\beta = 1.03$ and $\beta = 1.99$ were 4.5560×10^{-4} and 5.4430×10^{-4}. These point-wise errors are within the same range with the errors obtained using other RBF-MOLs like the MQ of[10] and IMQ, IQ, GIMQ of[13] as shown in Table 1.

Examples 4.2 is a nonhomogenous heat equation with a second order space derivative. To approximate the problem using RBF-MOLs, the GMQ RBF with $\beta = 1.03$ and $\beta = 1.99$ were used to discretize the space derivative and the domain was divided into $N = 31$ equally spaced centres. The spatial discretization was carried out using a single differentiation matrix since the problem is linear. The system of ODEs resulting from the spatial discretization of the PDE was integrated using the fourth order Runge-Kutta method. A time stepof $\Delta t = 5.0\times 10^{-7}$ was used to advance the approximation of the problem to a final time 1.0×10^{-4}. The shape parameters $\varepsilon = 1.0$ and $\varepsilon = 1.2$ were used for the GMQ with $\beta = 1.03$ and $\beta = 1.99$ respectively fornumerical approximations. The maximum point-wise errors obtained using the GMQ with $\beta = 1.03$ and $\beta = 1.99$ were 1.45×10^{-5} and 9.5×10^{-6} respectively. Comparing the point-wise errors of the nonstandard exponential values of the GMQ RBF-MOLs with the errors of the MQ, IMQ, IQ and GIMQ which are 2.5380×10^{-4}, 1.6330×10^{-4}, 7.4550×10^{-5} and 1.97800×10^{-5} (approximated under the same condition with shape parameters $\varepsilon = 5.0$, $\varepsilon = 5.0$, $\varepsilon = 3.0$ and $\varepsilon = 3.0$ for the respective RBF-MOLs) as shown in Table 2, we observed that the GMQ with $\beta = 1.99$ has the smallest point wise error, followed by the GMQ with $\beta = 1.03$.

Example 4.3 is the Burgers' equation with a space derivative of order two (2). The approximate solution of the problem was obtained using the GMQ RBF-MOLs with $\beta = 1.03$ and $\beta = 1.99$. The domain of the solution was divided into $N = 140$ equally spaced centres. Two separate differentiation matrices were used to carry out the spatial discretization since it is a nonlinear PDE. The system of ODEs that resulted from the spatial discretization were integrated using the explicit fourth order Runge-Kutta method. A time step size of $\Delta t = 1.0 \times 10^{-5}$ was chosen to advance the solution to a final time $FT = 1.2$. The results of the problem as obtained from the implementation of the MATLAB program are shown in Table 3. During the implementation of the MATLAB program, we observed that smaller values of the time step yielded a better approximation but took a long time to debug. Figures 5(a) and 6(a) represent the plots of the exact and approximate solutions. The values 80 and 110 were used as estimates of the shape parameters for the GMQ RBF-MOLs with $\beta = 1.03$ and $\beta = 1.99$ respectively. The maximum point wise errors for the GMQ RBF-MOLs with $\beta = 1.03$ and $\beta = 1.99$ are $3.57 \times 10^{-3}$and 3.57×10^{-3}as shown in Table 3. Comparing these errors with the point-wise errors of the MQ, IMQ, IQ and GIMQ RBF-MOLs as shown in Table 3, indicates that the nonstandard exponent values of the GMQ RBF-MOLs compared favourably with the rest of the RBF-MOLs for the Burgers' equation.

V Conclusion

Two GMQ RBF-MOLs with non-standard exponents, $\beta = 1.03$ and $\beta = 1.99$ were developed, implemented in MATLAB and utilized to approximate the solutions of some second order time-dependent PDEs in one space dimension. The numerical solutions were compared with the exact solutions and other RBF-MOLs, the approximate solutions showed that the non-standard RBF-MOLs method performed competitively with standard RBF-MOLs. Based on our results, we recommend that the GMQ RBF-MOLs with non-standard exponents be explored and applied to approximate other PDEs such as steady state PDEs and other methods of approximations that require RBFs, however, there is no clear evidence that these methods are better than the other RBF-MOLs.

References

[1]. Chen, W., Fu, Z.-J. and Chen, C. S.Recent advances in radial basis function collocation method. new york, springer heidelberg.2014.

[2]. Hardy, R. L. Multiquadric equations of topography and other irregular surfaces. Journal of Geographical Research. 1971;1905-1915.

[3]. Fasshauer, G. E. meshfree approximation methods with MATLAB, Singapore. World Scientific Publishing Co. Ltd.2007.

[4]. Harder, R. L. and Desmarais, R. N. Interpolation using surface splines. Journal of Aircraft.1972; 9(2): 189-191.

[5]. Meinguet, J. Basic mathematical aspects of surface spline interpolation: in Numerische Integration, G. Hammerlin (ed.), Birkhauser (Basel). 1972; 211- 220.

[6]. Franke, R. Scattered data interpolation: test of some methods. Mathematics of Computation. 1982; 38:181-200.

[7]. Kansa, E. J. Multiquadrics-a scattered data approximation scheme with application to computational fluid dynamics-I: surface approximations and surface estimates.Computer and Mathematics with Applications.1990a; 19(8-9): 127-145.

[8]. Kansa, E. J. Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics II: solutions to parabolic, hyperbolic and elliptic partial differential equations. Computers and Mathematics with Applications.1990b; 19(8-9): 147-161.

[9]. Kansa, E. J. and Holoborodko, P. On the Ill-nature of C^{∞} RBF strong collocation. Engineering Analysis with Boundary Elements. 2017; 78: 26-30.

[10]. Sarra, S. A. and Kansa, E. J. Multiquadric radial basis function approximation methods for the numerical solution of partial differential equations. California, Technical Science Press.2009.

[11]. Bibi, N. Meshless method of lines for numerical solutions of nonlinear time-dependent partial differential equations. Ph. D. Thesis, Ghulam Ishaq Khan Institute of Engineering Science and Technology, Topi, Swabi, Pakistan. 2011;139 pp.

[12]. Kansa, E. J. Explicit time integration of hyperbolic partial differential equations with mesh free radial basis functions. Engineering Analysis with Boundary Elements. 2007; 31:577-585.

[13]. Luga, T., Aboiyar, T. and Adee, S. O. Radial basis function methods for approximating time-dependent partial differential equations in one space dimension. Nigerian Journal of Pure and Applied Sciences. 2017a; 9(1):182-200.

[14]. Luga, T., Aboiyar, T. and Onah, E. S. Radial basis function methods for approximating the cubic nonlinear schrödinger equation in one space dimension. Journal of The Nigerian Association of Mathematical Physics2017b; 43: 325-332.

[15]. Wang, J. G. and Liu, G. R. On the optimal shape parameters of radial basis functions used for 2-D meshless methods. Computer Methods in Applied Mechanics and Engineering. 2002; 191: 2611-2630.

[16]. Xaio, J. R. and McCarthy, M. A. a local heaviside weighted meshless method for two-dimensional solids using radial basis functions. Computational Mechanics. 2003; 31: 301-315.

[17]. Chenoweth, M. A numerical study of generalized multiquadric radial basis function interpolation. Submitted to SIAM Undergraduate Research Online. 2009; 70pp.

[18]. Burden, R. L. and Faires, J. D. Numerical analysis. 9th edition, United States of America, Brooks/Cole. 2011; p.738.

Wahyuni Mustika. "Area of Outer Napoleon In The Parallelogram and Area of Outer Semi Napoleon in The Kite." IOSR Journal of Mathematics (IOSR-JM) 15.3 (2019): 38-46.