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Critical Depensation Growth function and Harvesting

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Abstract: *In this paper we have presented a PDE mathematical model. The population density depends on special location and time.In the equation, the first and the third terms deal with local behavior, whereas the second deals with horizontal redistribution. The population growth term follows critical depensation growth manner. The growth below its critical mass quantity is negative which shows biologically mate is rare so that population growth declines. The critical points origin and the carrying capacity are stable whereasthe critical mass quantity is unstable.*

To solidify the analytical results, numerical simulations are provided for hypothetical set of parametricvalues.

Keywords: *Critical depensation growth function, Diffusion, Population harvesting, Equilibrium solution, Stability analysis, Wave solution.*

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I. Introduction

The crabs population movement can be described as wave solution as explained by the Fisher in his equation [2] and the solution switches from the equilibrium state N^*=0 to the equilibrium state N^*= 1. Originally, the name king crab has been applied to a number of species, including the blue king crab, the Hanasaki king crab, the golden and scarlet king crabs, etc. as explained in [2]. As it is explained in [1] harvesting helps protect and defend marine reserves from endangered things. The researcher derived the yield maximizing spatial harvesting strategy in a specially explicit model in which no reserve are imposed. The model is a combination of the Schaefer harvest model(Schaefer 1957), which is at the foundation of many bio-economic analysis(Clerk 1990), and Fisher equation(Fisher 1937),a fundamental model in spatial populationdynamics(Kot2001).

In a critical depensation model [7], it is possible that the fish population level will be driven to extinction. If the depensation exists, fishery managers become upset because fished stocks may not recover after being over fished, even when fishing is stopped. The concept of Maximum Sustainable Yield (MSY) in general is required to practice the renewable resource management[Clark,1990]. This is not necessarily the best management method, because the long-run consumption profile does not coincide with that of utility maximization. The resource stock under the maximum sustainable yield(MSY) is not necessarily optimal with respect to production due to the positive relationship between productivity in harvesting activities and the resource stock size. In this paper we have modified the model [1] replacing the logistic population growth by critical depensation growth with carrying capacity K and critical mass quantity $K0$. The predicted solution of the mathematical model is well supported by the numerical simulation. Maximizing the yield crabs population or the fish population is also an important issue in this study

II. The critical depensation growth model

As it is explained well in [6], the population growth model which is written to the form

$$\frac{dN^*}{dT^*} = rN^*\left(1 - \frac{N^*}{K}\right)\left(\frac{N^*}{K_0} - 1\right) \qquad (1)$$

is strong critical depensation function. In the growth model (1), $N^*(T^*)$ represents the population size, r represents intrinsic growth rate of population, K is the carrying capacity of the population environment, K_0 is the critical mass quantity and $\frac{dN^*}{dT^*}$ is population growth rate. Below in Figure 1 the time series plot of the critical depensation growth model is plotted. In the figure we observe three equilibrium points N^*=0, N^*=K_0 and N^*= K. The first and the last critical points are stable whereas the middle is unstable. In Figure 2 we have growth curve of the critical depensation model. The curve is plotted for the population size function $N^*(T^*)$ versus the population growth rate function $\frac{dN^*}{dT^*}$. The growth rate is negative in the interval $(0,\ K_0)$ while it is positive in the interval $(\ K_0,\ K)$.

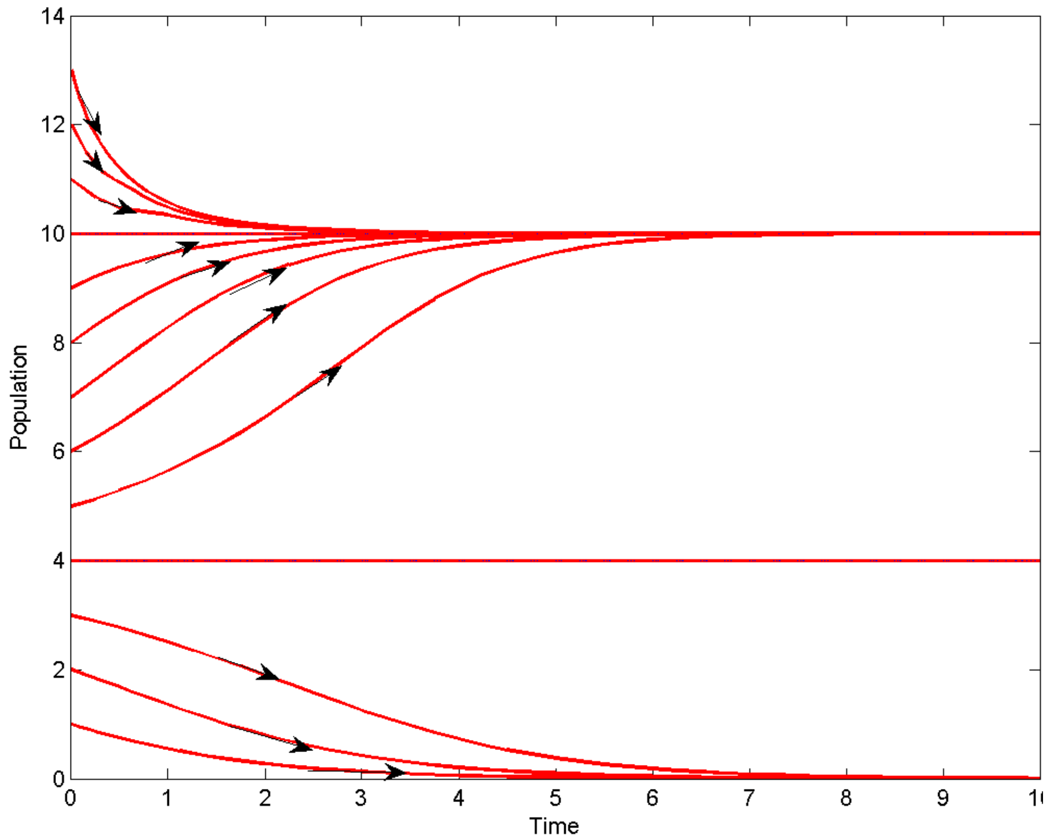

Fig1: Time series plot of critical depensation growth function for $r = 0.8, K_0 = 4, K = 10$

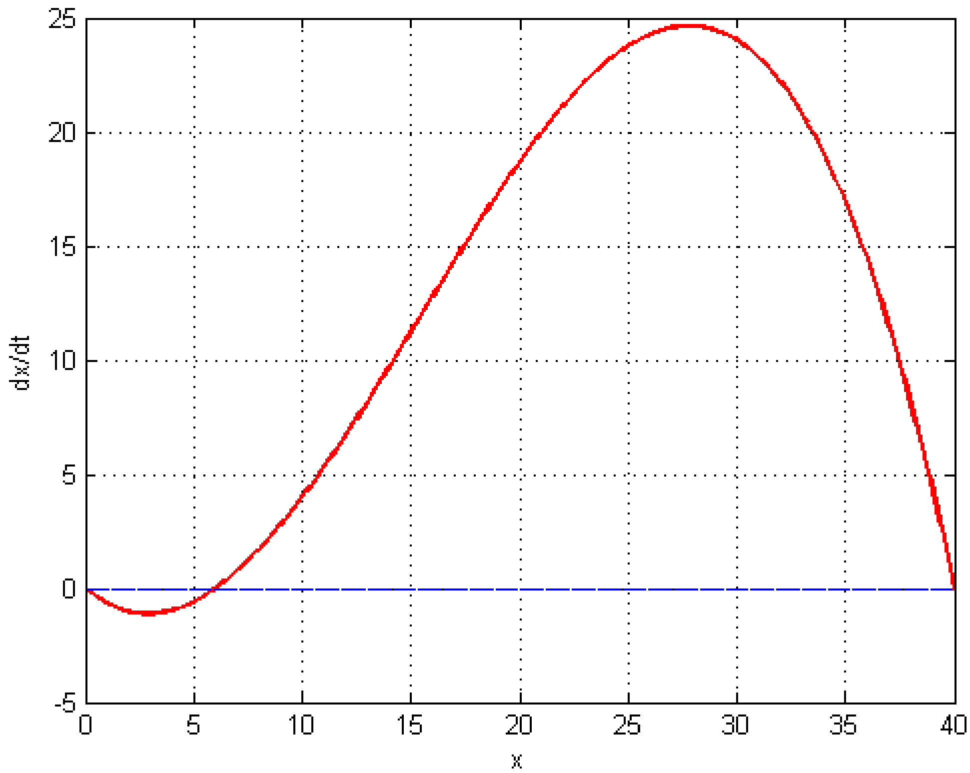

Figure 2: Growth curve of critical depensation growth function for $r = 0.8, K_0 = 6, K = 40$

III. The Extended Mathematical Model

For the population density $N^*(X^*, T^*)$ that depends on the spatial location $X^*$and time T^*, N^* is directed by the equation

$$\frac{\partial N^*}{\partial T^*} = RN^*\left(1-\frac{N^*}{K}\right)\left(\frac{N^*}{K_0}-1\right) + D\frac{\partial^2 N^8}{\partial X^{*2}} - QE^*(X^*)\,N^*, \qquad (2)$$

$$0 < X^* < L, N^*(0, T^*) = 0,\; N^*(L, T^*) = 0 \qquad (3)$$

Where $E^* = E^*(X^*)$ is the effort located at location X^*. The model as interpreted in [1] states that the change per unit time of the population N^* at a given location X^* and time T^* is controlled by the critical depensation population growth function, the movement of the population by diffusion and a Harvesting term. In the population growth, R is intrinsic growth rate of the population, K is the environmental carrying capacity and K_0 is the critical mass quantity where $K_0 < K$. In the diffusion term, the diffusion coefficient has dimension $[D] = m^2/s$, and in the harvesting term the harvesting rate at a given location is proportional to the product of the stock size N^* and the effort located at that location $E^*(X^*)$ where Q is the proportionality constant known as 'catch ability' coefficient. The location of the habitat length and the boundary condition are parts of the model. The model describes a population living in a patch of suitable habitat, of length L, surrounded by unsuitable habitat.

The equilibrium yield is given by the integral equation [1],

$$N^*(X^*, T^*) = \int_0^L QE^*(X^*)N^*(X^*)dX^* \qquad (4)$$

$$0 \le E^* \le E^*_{Max} \qquad (5)$$

Eq'n (5) is because of limitation on the population effort.

The equilibrium yield is given by the integral equation [1],

$$Y^*(N^*, E^*) = \int_0^L Q\,E^*(X^*)N^*(X^*)dX^*.$$

Scaling the mathematical model

Time Scale: $T^* = \frac{1}{R}t$, Population Size Scale: $N^* = KN$, Length Scale: $X^* = Lx$, Harvesting Effort Scale: $E^* = b(x)\frac{R}{Q}$, The Yield Scale: $Y^* = (K\sqrt{RD})Y$.

Assuming $\beta = \frac{K_0}{K}, \sigma = \frac{D}{RL^2}$, the final scaled equation has the form

$$\frac{\partial N}{\partial t} = N(1-N)\left(\frac{N}{\beta}-1\right) + \sigma\frac{\partial^2 N^8}{\partial X^2} - b(x)N, 0 < x < 1, \qquad (6)$$

$$N(0, t) = 0,\; N(1, t) = 0 \qquad (7)$$

The scaled equilibrium yield also looks

$$Y(N, b) = \frac{1}{\sqrt{\sigma}}\int_0^1 b(x)\,N(x)dx,\; 0 \le b \le b_{max}$$

Here we observe that the scaled equation contains three dimensionless parameters but the dimensioned equation has seven parameters.

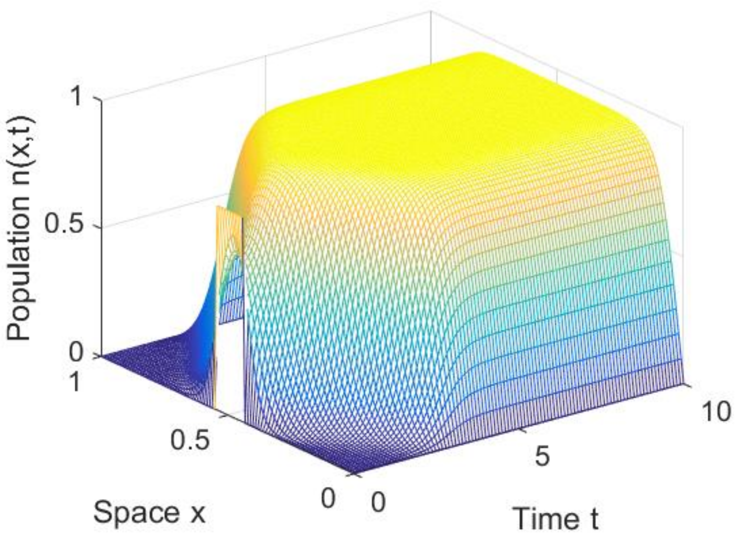

Fig 3 is the numerical solution of the model where $\beta = 0.1$, $\sigma = 0.01, b = 0.5$.

IV. Some Comments About The Optimal Catch Problem

4.1 The population growth is logistic

Habitat is one dimensional, $0 \leq x \leq 1$.
In this case, there is a limit on how large the diffusion coefficient may be.
With too large diffusion the habitat is empty. The problem consists of maximization of the y

$$\boldsymbol{Y(\, b) = \int_0^1 b(x)\, N(x) dx,}$$

by finding an optimal $\boldsymbol{b(x)}$ for $\boldsymbol{0} < b(x) < \boldsymbol{b_{max}}$.
The full equation is

$$\boldsymbol{\frac{\partial N}{\partial t} = N(1-N)\left(\frac{N}{\beta}-1\right) + \sigma \frac{\partial^2 N^8}{\partial X^{*2}} - b(x)N},$$

with the boundary conditions

$$\boldsymbol{N(\, 0,\ t) = 0,\ N(\, 1, t) = 0.}$$

It is important to observe N and b are mutually dependent. Our interest deals with the stationary situation where

$$\boldsymbol{N(1-N)\left(\frac{N}{\beta}-1\right) + \sigma \frac{\partial^2 N^8}{\partial X^2} - b(x)N = 0},$$

$$\boldsymbol{N(\, 0,) = 0,\ N(\, 1) = 0}$$

The first and the third terms deal with local behavior, whereas the second deals with horizontal redistribution. We notice the special cases $\sigma \to 0$ and $\sigma \to \infty$. The first one considers a situation with the habitat consisting of vertical "cells". The cell walls stop the horizontal diffusion; as a result and we may consider each cell separately,

$$\boldsymbol{N(1-N)\left(\frac{N}{\beta}-1\right) - bN = 0.}$$

The stationary points

$$N_0 = \begin{cases} 1-b\ for\ b < 1, \\ \ \ 0\ for\ b \geq 1. \end{cases}$$

Stationery points do not depend on time and transcritical bifurcation occurs at $b = 1$. Thus, the local yield is $Y = (1-\mathrm{b})b$.
The maximal local sustainable yield occurs for $b = \frac{1}{2}$ and the maximum sustainable yield is

$$Y_{max} = \left(1-\frac{1}{2}\right)\frac{1}{2} = \frac{1}{4}.$$

4. The population growth is critical depensation

Habitat is one dimensional, $0 \leq x \leq 1$.
In this case, there is also a limit on how large the diffusion coefficient may be.
With too large diffusion the habitat is emptied. The problem also consists of maximization of the yield:

$$\boldsymbol{Y(\, b) = \int_0^1 b(x)\, N(x) dx,}$$

by finding an optimal $b(x)$ for $\boldsymbol{0} < b(x) \leq \boldsymbol{b_{max}}$.
The full equation is

$$\boldsymbol{\frac{\partial N}{\partial t} = N(1-N)\left(\frac{N}{\beta}-1\right) + \sigma \frac{\partial^2 N^8}{\partial X^{*2}} - bN},$$

with the boundary conditions

$$\boldsymbol{N(\, 0,\ t) = 0,\ N(\, 1, t) = 0.}$$

It is important to observe that N and b are mutually dependent. Our interest deals with the stationary solutions where

$$\boldsymbol{N(1-N)\left(\frac{N}{\beta}-1\right) + \sigma \frac{\partial^2 N^8}{\partial X^{*2}} - bN = 0,}$$

$$\boldsymbol{N(\, 0,) = 0,\ N(\, 1) = 0.}$$

The first and the third terms deal with local behavior, whereas the second term deals with horizontal redistribution. We notice the special cases $\sigma \to 0$ and $\sigma \to \infty$. The first one considers a situation with the habitat consisting of vertical "cells". The cell walls stop the horizontal diffusion and we may consider each cell separately,

$$\boldsymbol{N(1-N)\left(\frac{N}{\beta}-1\right) - bN = 0\ .}$$

Here, $N_0 = 0$, $N_1 = \frac{(\beta+1)-\sqrt{(\beta+1)^2-4\beta(1+b)}}{2}$ and $N_2 = \frac{(\beta+1)+\sqrt{(\beta+1)^2-4\beta(1+b)}}{2}$ are stationary points. The equilibrium points N_1 and N_2 exist if $\left(\frac{\beta+1}{2\sqrt{\beta}}\right)^2 \geq (1+b)$.

The equilibrium points N_0 and N_2 are stable while N_1 is unstable. The local yield at the equilibrium situation is

$$Y = N_2 b = \left(\frac{(\beta+1) + \sqrt{(\beta+1)^2 - 4\beta(1+b)}}{2} \right) b$$

Since $\frac{dY}{db} = 0$ give $b = 0$ which is not positive real, the maximum sustainable yield is occurred at the end point, b_{max} and hence it follows that the maximum sustainable yield, Y_{MSY} is

$$Y_{MSY} = b_{max} \left(\frac{(\beta+1)+\sqrt{(\beta+1)^2-4\beta(1+b_{max})}}{2} \right), \left(\frac{\beta+1}{2\sqrt{\beta}} \right)^2 \geq (1 + b_{max}).$$

Furthermore, from the equation

$$\boldsymbol{N(1-N)\left(\frac{N}{\beta}-1\right) - bN = 0}$$

We have

$$\boldsymbol{b = (1-N)\left(\frac{N}{\beta}-1\right)}$$

which is a concave function of N.
Maximum of the concave function or parabola: From above, we observe immediately that

$$\boldsymbol{\frac{db}{dN} = 0}$$

implies that

$$\boldsymbol{N_{max} = \frac{\beta+1}{2}, \ b_{max} = \frac{(1-\beta)^2}{4\beta}.}$$

The bifurcation point in the bN- plane is $\left(\frac{(1-\beta)^2}{4\beta}, \frac{\beta+1}{2}\right)$ where stability change is observed. The limit behavior when $\sigma \to \infty$. is the zero-solution. The bifurcation diagram, plot of N over b, for $\beta = 0.1$ is as follows, and the bifurcation point in this case is $(2.205, 0.55)$.

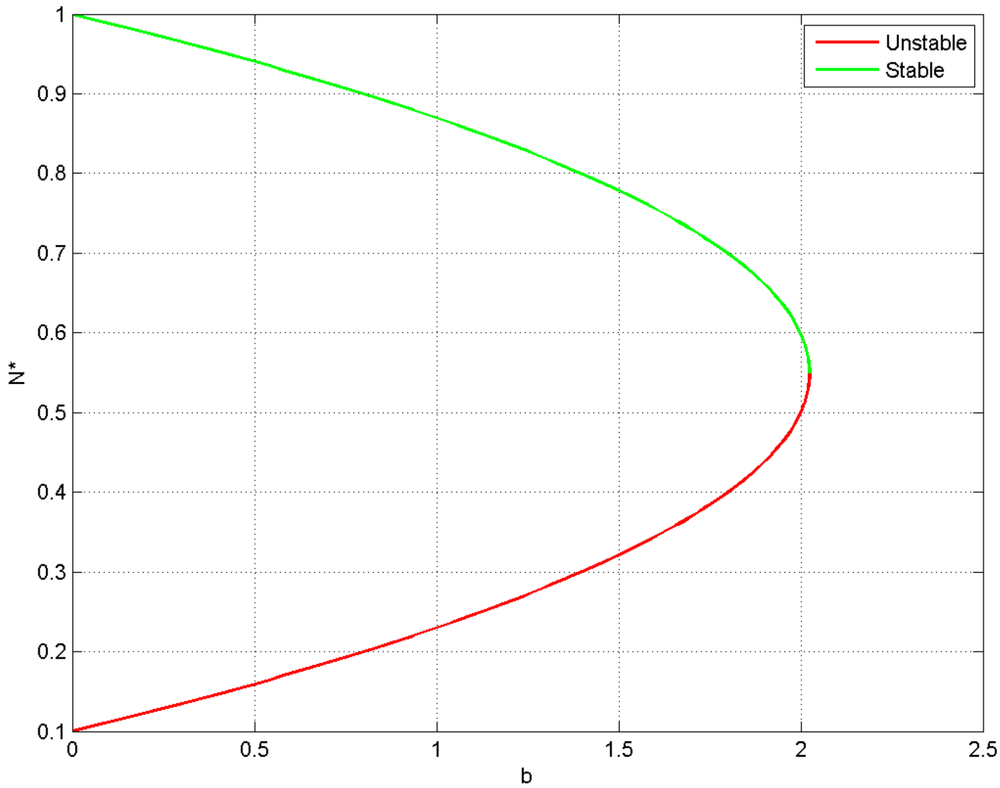

Figure 4: The bifurcation diagram for $\beta = 0.1$.

The graph shows the dependency of the equilibrial level of the population on the population mortality. The red part of the graph is unstable whereas the green part is stable.

4.2.1 Suppose the harvesting term is zero

In this case the mathematical model has the form

$$\frac{\partial N}{\partial t} = N(1-N)\left(\frac{N}{\beta}-1\right) + \frac{\partial^2 N^8}{\partial X^2},\ 0 < x < 1 \qquad (8)$$

$$N(0,\ t) = 0,\ N(1, t) = 0. \qquad (9)$$

The numerical solution can be treated here for the case while harvesting term is zero. If $N = 2\beta$, equation (8) becomes Fisher's equation.

4.2.2 Wave solution

We can rewrite equation (8) as

$$\frac{\partial N}{\partial t} = \left(\frac{1}{\beta}\right) N(1-N)(N-\beta) + \sigma\,\frac{\partial^2 N^8}{\partial X^2}, \qquad (10)$$

Equation (10) is the Nagumo(Bi-stable) Equation. We can look traveling wave solution of the form [2]

$$u(x,t) = u(x-ct) = u(z),$$

where $c = (1-2\beta)\sqrt{\frac{\sigma}{2\beta}}$ which is the velocity that the density of the population travels.

Remarks:

i) The wave advances for $0 < \beta < 0.5$ and it retreats for $0.5 < \beta < 1$.

ii) Alternatively $u(x,t) = u(x-ct) = u(z) = \frac{1}{1-exp\left(\frac{z}{\sqrt{2\sigma\beta}}\right)}$,

4.3 Equilibrium points and stability analysis

4.3.1 The equilibrium points

The population is at equilibrium if $\frac{\partial N}{\partial t} = 0$ and this implies

$$N(1-N)\left(\frac{N}{\beta}-1\right) + \sigma\,\frac{\partial^2 N^8}{\partial X^2},\ = 0,\ 0 < x < 1 \qquad (11)$$

$$N(0) = 0,\ N(1) = 0 \qquad (12)$$

Setting $\frac{dN}{dx} = v$ we have the following system of differential equations with boundary condition as

$$\frac{dN}{dx} = v, \qquad (13)$$

$$\frac{dv}{dx} = \frac{1}{\sigma} N(N-1)\left(\frac{N}{\beta}-1\right), \qquad (14)$$

$$N(0) = 0, N(1) = 0, v(0) = 0, v(1) = 0 \qquad (15)$$

This ODE system has three phase plane equilibria: $(0,0)$, $(1,0)$, and $(\beta, 0)$.

4.3.2 Local stability analysis

The Jacobin matrix $J(N, v)$ of the system is

$$J(N,v) = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

where $A = \frac{1}{\sigma}\left(\frac{3}{\beta}n^2 - 2\left(\frac{\beta+1}{\beta}\right)n + 1\right)$. The Jacobin matrix evaluated at $(0,0)$ is

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{\sigma} & 0 \end{pmatrix},$$

and hence, the trivial equilibrium point is saddle point which is unstable.

In similar manner, the Jacobin matrix evaluated at $(1,0)$ is

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ \frac{1-\beta}{\sigma\beta} & 0 \end{pmatrix},$$

and hence, the equilibrium point is saddle point which is also unstable.

The Jacobin matrix evaluated at $(\beta, 0)$ is

$$J(\beta,0) = \begin{pmatrix} 0 & 1 \\ 3\frac{\beta}{\sigma} - 2\beta - 1 & 0 \end{pmatrix},$$

and this equilibrium point is a center for the linearized system since the expression $(3\frac{\beta}{\sigma} - 2\beta - 1)$ is a positive quantity. For this to be sure more we do the following as it is done in [5]. Equation (10) has first integral and multiplying (10) through by $N' = \frac{dN}{dx}$,

$$\sigma\frac{d^2N}{dx^2}N' + \left(\frac{1}{\beta}\right)N(1-N)(N-\beta)N' = 0 \qquad (16)$$

Integrating this with respect to *x* produces

$$\frac{\sigma}{2}(N')^2+\frac{N^3}{3}(1+\beta)-\beta\left(\frac{N^2}{2}\right)-\frac{N^4}{4}=c. \qquad (17)$$

This can be also written as

$$\frac{\sigma}{2}v^2+\frac{N^3}{3}(1+\beta)-\beta\left(\frac{N^2}{2}\right)-\frac{N^4}{4}=c. \qquad (18)$$

and the label curves are orbits in the phase plane as depicted in Fig5 below.

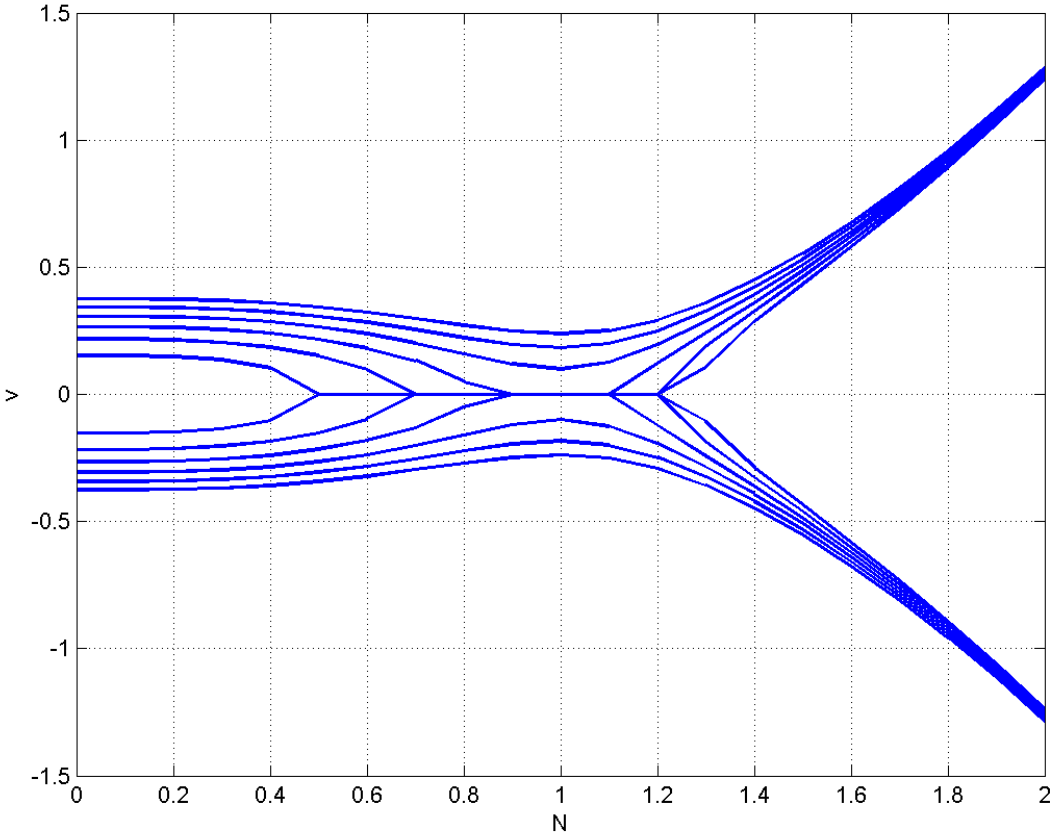

Figure 5: Level curves for $\beta = 0.07, \sigma = 1.7$

V. Result and discussion

The scaled equation contains three dimensionless parameters whereas the dimensioned equation has seven parameters. The population growth termis critical depensation growth function. The growth below its critical mass quantity is negative which shows sexual mate is small so that population growth declines. The critical points origin and the carrying capacity are stable while the critical mass quantity is unstable. To solidify the analyticalresults, numerical simulations are provided for hypothetical set of parametric values. Some of the comments observed on the optimal catch problem are:

i) The population growth is logistic: The first and the third terms deal with local behavior, whereas the second deals with horizontal redistribution. We observed two special cases $\sigma \to 0$ and $\sigma \to \infty$. The first one considers a situation with the habitat consisting of vertical "cells". The cell walls stop the horizontal diffusion and in this case, we found out the stationary points are

$$N_0 = \begin{cases} 1-b\ for\ b<1, \\ 0\ for\ b \geq 1. \end{cases}$$

I identified the transcritical bifurcation occurs at $b=1$ the maximal local sustainable yield occurs for $b=\frac{1}{2}$ and the maximum sustainable yield is

$$Y_{max} = \left(1-\frac{1}{2}\right)\frac{1}{2}=\frac{1}{4}.$$

The limit behavior when $\sigma \to \infty$, i.e., the diffusion is high and the habitat is emptied, is the zero solution.

ii) The population growth is critical depensation:

Similarly, if $\sigma \to 0$, the first one considers a situation with the habitat consisting of vertical "cells". Hence, I found out stationary points

$N_0 = 0,\ N_1 = \frac{(\beta+1)-\sqrt{(\beta+1)^2-4\beta(1+b)}}{2}$ and $N_2 = \frac{(\beta+1)+\sqrt{(\beta+1)^2-4\beta(1+b)}}{2}$.

The stationery points N_0 and N_1 are stable while N_3 is unstable.The maximum sustainable yield in the equilibrium situation is found as $Y_{MSY} = b_{max}\left(\frac{(\beta+1)+\sqrt{(\beta+1)^2-4\beta(1+b_{max})}}{2}\right), \left(\frac{\beta+1}{2\sqrt{\beta}}\right)^2 \geq (1+b_{max})$.As depicted, Fig 3 is the numerical solution of the mathematical model for $\beta = 0.1,\ \sigma = 0.01, b = 0.5$ and Fig 4 shows the dependency of the equilibrial level of the population on the population mortality. I also considered the case harvesting term is zero and I found out the traveling wave solution, and I justified the axial equilibrium point $(\beta, 0)$ is really a center, as Fig5 justifies this fact.

VI. Conclusion

Here, in this paper, the PDE mathematical model equation contain: the population growth term, diffusion term and the harvesting term. If the diffusion term $\sigma \to 0$ we have a situation with the habitat consisting of vertical "cells". Accordingly the stationary points and the maximum sustainable yield are found. It is also verified the dependency of the equilibria on the control parameter b. The research also did the qualitative analysis of the model equation for different cases, especially traveling wave solution is found if the harvesting term is zero

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