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Proving Riemann Hypothesis by Lagarias's Equivalent

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Abstract: *One of the most elusive unsolved problems of today is Riemann hypothesis. For long mathematicians have struggled to prove this problem,and also tried to devise an elementary version of the problem, proof of which indirectly proves Riemann hypothesis. In 2002 J. C. Lagarias published such an elementary version of the hypothesis which has been widely accepted as an elementary equivalent of Riemann hypothesis. This article attempts to prove Lagarias's condition which consequently proves Riemann hypothesis.*
Key words: *Harmonic number, natural logarithm, factorial.*

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I. Introduction

In mathematics, the **Riemann hypothesis**[1]is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. [2]It is of great interest in number theory because it implies results about the distribution of prime numbers. It was proposed by Bernhard Riemann (1859), after whom it is named. German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function

$\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \ldots$

called the *Riemann Zeta function*. The Riemann hypothesis asserts that all *interesting* solutions of the equation

$\zeta(s) = 0$

lie on a certain vertical straight line.
This has been checked for the first 10,000,000,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers. [3]

II. Equivalents of Riemann hypothesis:

In 1984 Guy Robin has showedthat ,
$\sigma(n)= e^{\gamma} n \, loglog(n)$ for all $n \geq 5041$
The problem is a necessary and sufficient condition for the Riemann hypothesis. Here γ=0.57721 is the Euler–Mascheroni constant and $\sigma(n)$is the sum of divisors of the positive integer n, given by
$\sigma(n)=\sum_{d\,|\,n} d$
Building on this, Jeffrey Lagariasshowed the equivalence of the Riemann hypothesis to a condition on harmonic sums[4] [5], namely
$\sigma(n) \leq H_n + e^{Hn} \; lnH_n$
Here, $H_n$is the n-th harmonic number equal to the sum of the reciprocals of the first n positive integers
$H_n=\sum_{k=1}^{n} \frac{1}{k}$.

III. Proof of Lagarias's relation:

Proof: The proposition to prove is that,
$\sigma(n) \leq H_n + e^{H_n} \text{In} H_n$
When,nth harmonic number[6],
$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$
$\sigma(n) = 1 + f_1 + f_2 + f_3 \ldots. \ldots.n$
Where f_1, f_2, f_3.... .... are factors of n
We see,
$\sigma(4) = 1+2+4$

$=4(\frac{1}{4}+\frac{1}{2}+1)$
$=4\ (1+\frac{1}{2}+\frac{1}{4})$
σ(6)= 1+2+3+6
$=6(\frac{1}{6}+\frac{1}{3}+\frac{1}{2}+1)$
$=6\ (1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6})$

So,
$\sigma(n)= 1+f_1+f_2+f_3.... \ ....n$
$=n(\frac{1}{n}...+\frac{1}{f3}+\frac{1}{f2}+\frac{1}{f1}+1)$
$=n(1+\frac{1}{f1}+\frac{1}{f2}+\frac{1}{f3}...+\frac{1}{n})$
If $H_\sigma(n)$=$1+\frac{1}{f1}+\frac{1}{f2}+\frac{1}{f3}...+\frac{1}{n}$let its complementary to harmonic series is $H'_\sigma(n)$so that,
$H_\sigma(n)+ H'_\sigma(n) = H_n$
For example,
$H_\sigma(4)+ H'_\sigma(4) =(1+\frac{1}{2}+\frac{1}{4})+(\frac{1}{3}) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}= H_4$
$H_\sigma(6)+ H'_\sigma(6) =(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6})+(\frac{1}{4}+\frac{1}{5}) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}= H_6$
$H_\sigma(15) + H'_\sigma(15) =(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{15})+(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}= H_{15}$
So, $\sigma(n)= 1+f_1+f_2+f_3.... \ ....n$
$=n(1+\frac{1}{f1}+\frac{1}{f2}+\frac{1}{f3}...+\frac{1}{n})$
$= nH_\sigma(n) = n\ (H_n - H'_\sigma(n))= n\ H_n(1-\frac{H'_\sigma(n)}{H_n})$.
It is quite evident from a number as little as 15 that for larger numbers, $n>>>1$, there would be $H'_\sigma(n)>>>H_\sigma(n)$for most n with several factors, more so when n is a prime.
e.g. $H_\sigma(p) = 1+\frac{1}{p}$ and $H'_\sigma(p) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}...+\frac{1}{p-1}$
$H'_\sigma(n)>H_\sigma(n)$is true even in case of n=larger primorials, as primes are sparse then.
Reverse is true in case of n=smaller primorialor,n= smaller factorial; in these cases $H'_\sigma(n)<H_\sigma(n)$.
e.g. 2!=4 and $H_\sigma(4) = 1+\frac{1}{2}+\frac{1}{4}$ and $H'_\sigma(4) = \frac{1}{3}$
3!=6 and $H_\sigma(6) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$ and $H'_\sigma(6) = \frac{1}{4}+\frac{1}{5}$ here, $H'_\sigma(n)<H_\sigma(n)$.
But even for the next bigger factorial numbers,
4!=24 and $H_\sigma(24) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{24}$ and $H'_\sigma(24) = \frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{20}+\frac{1}{21}+\frac{1}{22}+\frac{1}{23}$
5!=120 and $H_\sigma(120) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{20}+\frac{1}{24}+\frac{1}{30}+\frac{1}{60}+\frac{1}{120}$and $H'_\sigma(120) = \frac{1}{7}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{21}+\frac{1}{22}+\frac{1}{23}+\frac{1}{25}+\frac{1}{26}+\frac{1}{27}+\frac{1}{28}+\frac{1}{29}+\frac{1}{31}+\frac{1}{32}+\frac{1}{33}+\frac{1}{34}+\frac{1}{35}+\frac{1}{36}+\frac{1}{37}+\frac{1}{38}+\frac{1}{39}+\frac{1}{40}+\frac{1}{41}+\frac{1}{42}+\frac{1}{43}+\frac{1}{44}+\frac{1}{45}+\frac{1}{46}\dots\dots\dots+\frac{1}{59}+\frac{1}{61}\dots\dots\dots.+\frac{1}{114}+\frac{1}{115}+\frac{1}{116}+\frac{1}{117}+\frac{1}{118}+\frac{1}{119}$.
It is becoming $H'_\sigma(n)>H_\sigma(n)$when$n$ is large enough and$n$=m! and (n,m)∈ $\mathbb{Z}^+$
When n=a^m and (n,a,m)∈ $\mathbb{Z}^+$
$H_\sigma(n) = 1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}\dots\dots+\frac{1}{a^m}$and $H'_\sigma(n) = \frac{1}{a+1}+\frac{1}{a+2}+\frac{1}{a+3}\dots\dots+\frac{1}{a^2-1}+\frac{1}{a^2+1}+\frac{1}{a^2+2}+\frac{1}{a^2+3}\dots\dots+\frac{1}{a^3-1}\dots\dots+\frac{1}{a^m-1}$.
It is becoming $H'_\sigma(n)>H_\sigma(n)$when n is large enough and n=a^m and (n,a,m)∈ $\mathbb{Z}^+$
As it has been proposed,
$\sigma(n)\leq H_n + e^{H_n}\text{In}H_n$
or, $n\ H_n(1-\frac{H'_\sigma(n)}{H_n})\leq H_n + e^{H_n}\text{In}H_n$
or, $n\ H_n(1-\frac{H'_\sigma(n)}{H_n}) - H_n\leq e^{H_n}\text{In}H_n$
or, $n\ H_n(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\leq e^{H_n}\text{In}H_n$
or, $H_n(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\leq \frac{1}{n}e^{H_n}\text{In}H_n$

or, $H_n(1-\frac{H'_\sigma(n)}{Hn}-\frac{1}{n})\leq e^{H_n}\text{In}H_n^{\frac{1}{n}}$
Let, $H_n=e^x$ when *x* is a positive real number.
So, $H_n(1-\frac{H'_\sigma(n)}{Hn}-\frac{1}{n})\leq e^{H_n}\text{In}(e^x)^{\frac{1}{n}}$
or,$H_n(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\leq e^{H_n}\text{In}e^{\frac{x}{n}}$
So, $H_n(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\leq e^{H_n}\frac{x}{n}$
We know, $e^1=e=2.71828182845904523536028747135 27 \ldots > 1$,
$e^2=7.389056098930650227230427460575\ldots > 2, e^3=20.085536923187667740928529654582\ldots > 3$,
thus,$e^x > x$ and $e^{H_n} > H_n$,and when $n>>1$ then, $e^{Hn} >> H_n$ and $e^x >> x$
Again,$H_1=\frac{1}{1}=1$, $H_2=1+\frac{1}{2}<2$, $H_3=1+\frac{1}{2}+\frac{1}{3}<3$, thus $H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}<n$
So, $H_n=e^x<n$ and, $x\ll n$ or, $\frac{x}{n}\ll 1$ so, $0<\frac{x}{n}\ll 1$
Quite clearly, for most numbers ,$(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\ll 1$, and even for factorials and small primorials, $(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})<1$
For a larger number,whichare not factorial or small primorialhaving a few or several factors, when $H'_\sigma(n)>>>H\sigma(n)$,
$1=\frac{H_n}{H_n}=\frac{H'_\sigma(n)+H_\sigma(n)}{H_n}\rightarrow\frac{H'_\sigma(n)}{H_n}$
Conversely, $\frac{H'_\sigma(n)}{H_n}\rightarrow 1$ and $(1-\frac{H'_\sigma(n)}{H_n})\rightarrow 0$ when $n>>>1$ or, more precisely, when $n\rightarrow\infty$, $\frac{1}{n}\rightarrow 0$.
So, for most numbers ,$(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\ll 1$
So, in the equation,$H_n(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\leq e^{H_n}\frac{x}{n}$it has been proved that,
$e^{Hn} >> H_n$or, $H_n\ll e^{Hn}$ that means $x\ll e^x$.
$(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})<1$ on the left hand side, $0<\frac{x}{n}<1$ on the right hand side.
So, when *n* has a few or several factors,
$H_n(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\leq e^{H_n}\frac{x}{n}$.
Now, to get a more general picture, we further modify the relation, by placing$H_n=e^x$, as per our assumption,
$H_n(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\leq e^{H_n}\frac{x}{n}$
or,$e^x(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n})\leq e^{e^x}\frac{x}{n}$
or, $\ln[e^x(1-\frac{H'_\sigma(n))}{H_n}-\frac{1}{n})]\leq\ln[e^{e^x}\frac{x}{n}]$
or, $x+\ln\left(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n}\right)\leq e^x+\ln\frac{x}{n}$
Here, $x<e^x$and for larger numbers $x\ll e^x$
We got, $\left(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n}\right)<1$ and $\frac{x}{n}<1$
Then, $\ln\left(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n}\right)<1$ and $\ln\frac{x}{n}<1$
As $n>>H_n\geq H'_\sigma(n)>1$ so, $\frac{H'_\sigma(n)}{H_n}>\frac{1}{n}$. Also,$\ln\left(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n}\right)$ will vary more, more negative in case of large *n* and$\left(1-(\frac{H'_\sigma(n)}{H_n}+\frac{1}{n})\right)\rightarrow 0$with a few factors reducing the left hand side of our given relation, less so in case of *n* with many factors compared to its value and$\left(1-(\frac{H'_\sigma(n)}{H_n}+\frac{1}{n})\right)\rightarrow 1$as in case of factorials,in any way subtracting from the left hand side, $\ln\left(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n}\right)=\ln\left(1-(\frac{H'_\sigma(n)}{H_n}+\frac{1}{n})\right)$will always be negative.
On the right hand side, $e^x+\ln\frac{x}{n}=H_n+\ln x-\ln n=(H_n-\ln n)+\ln x$
We know, $\lim_{n\rightarrow\infty}\quad(H_n-\ln n)\rightarrow\gamma$so, $H_n-\ln n=\gamma+k$
Here, Euler–Mascheroni constant, $\gamma=H_n-\ln n-\frac{1}{2n}+\frac{1}{12n^2}-\frac{1}{120n^4}\ldots\ldots..=H_n-\ln n-k$.
$k=\quad\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}\ldots\ldots..$, and $\gamma\sim 0.57721$.[6]
Therefore, both $(H_n-\ln n)$ and $\ln x$is positive
So, $x+\ln\left(1-\frac{H'_\sigma(n)}{H_n}-\frac{1}{n}\right)\leq e^x+\ln\frac{x}{n}$

So, $x + \ln\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right) \le e^x + \ln\frac{x}{n}$(step J)
There could be several situations,

1) When n is a prime, p_n or a number with a few or several factors, $H'_\sigma(n) \gg H_\sigma(n)$, $H'_\sigma(n) \to H_n$ and $\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right)$ is a very small fraction, and $\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right) \to 0$ when $n \to \infty$.

2) When n is a smallprimorial, $p_n\#$ or a small factorial, $m!$, $H_\sigma(n) \gg H'_\sigma(n)$, $H_\sigma(n) \to H_n$, and $\Big(1 -$ *(H′σnHn+1n)* is a bigger fraction, and *1−(H′σnHn+1n)→1.*

3) When n is a larger primorial, $p_n\#$ or a larger factorial, $m!$, or a power of an integer $n=a^m$ or a larger integer with a significant number of factors, $H_\sigma(n) > H'_\sigma(n)$,
and $\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right)$is between 0 and 1.

4) Our quasi-theoretical situation when $H_\sigma(n) = H_n$ and $H'_\sigma(n) = 0$. Only actual example is when n=2, then, $H_\sigma(n) = 1 + \frac{1}{2} = H_n$, $H'_\sigma(n) = 0$. In this case, $\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right) = \left(1 - \frac{1}{n}\right)$ and when, theoretically, $n \to \infty$ then, $\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right) = 1$. This is the situation when the left hand side of our relation (step J) takes the greatest value, and our relation (step J) is least likely to be true. We will try to prove that our relation (step J) is true even in this case, consequently, it will be true for all other cases.

Here, $x < e^x$, and except for a few initial numbers, $x \ll e^x$.

<u>For situation 1),</u> $\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right)$ is a very small fraction, and $\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right) \to \frac{1}{n} \to 0$ when $n \to \infty$ and $\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right) \le \frac{x}{n}$, then,

$$x + \ln\left(1 - (\frac{H'_\sigma(n)}{H_n} + \frac{1}{n})\right) \le e^x + \ln\frac{x}{n}$$

<u>For situation 4),</u> when the left hand side takes highest value,

$$x + \ln\left(1 - \frac{1}{n}\right) \le e^x + \ln\frac{x}{n}$$

or, $x + \ln\left(\frac{n-1}{n}\right) \le e^x + \ln\frac{x}{n}$
or, $x + \ln\left(\frac{n-1}{n}\right) - \ln\frac{x}{n} \le e^x$
or, $x + \ln\left(\frac{n-1}{n}\right) + \ln\frac{n}{x} \le e^x$

Now we see the first few harmonic numbers and examples of values of terms of the relation above (Table 1).

Table 1(some inaccuracies likely due to rounding up the figures)

n	H_n=e^x	$x = \ln e^x = \ln H_n$	$\frac{n-1}{n}$	$\ln\left(\frac{n-1}{n}\right)$	$\frac{n}{x}$	$\ln\frac{n}{x}$	$\ln n$
1	1	0	0	-∞	∞		0
2	1.5	~0.40547	0.5	-0.693147	4.9325	1.5958	0.6931
3	~1.83333	~0.60613	0.666666	-0.405465	4.9494	1.5992	
4	~2.08333	~0.73396	0.75	-0.287682			
5	~2.28333	~0.82563	~0.85714	-0.154154			
6	2.45	~0.89608					
7	~2.59286	~0.95276					
8	~2.71786	~0.99984					
9	~2.82897	~1.03991					
10	~2.92897	~1.07465	0.9	-0.10536	9.30535	2.2305	2.3025
⋮	⋮	⋮					
20	~3.59774	~1.28031	0.95	-0.0512932	15.6212	2.7486	2.9957
⋮	⋮	⋮					
30	~3.99499	~1.38504					
⋮	⋮	⋮					
40	~4.27854	~1.45361					
⋮	⋮	⋮					
50	~4.49921	~1.50390					
⋮	⋮	⋮					
130	~5.44859	~1.69536					
⋮	⋮	⋮					
150	~5.59118	~1.72119					
⋮	⋮	⋮					
200	~5.87803	~1.77122					
⋮	⋮	⋮					
250	~6.10068	~1.80840					
⋮	⋮	⋮					

300	~6.28266	~1.83779					
⋮	⋮	⋮					
400	~6.56993	~1.88250					
⋮	⋮	⋮					
500	~6.79282	~1.91587			260.978	5.5644	6.2146
⋮	⋮	⋮					
900	~7.38017	~1.99880	~0.9988	-0.0011206			
⋮	⋮	⋮					
925	~7.40755	~2.00250					
⋮	⋮	⋮					
10000	~9.78761	~2.28112	0.9999	-0.0001	4383.81	8.3856	9.21034
⋮	⋮	⋮					
1000000	~14.3927	~2.66672	0.999999	-0.000001	374992.5	12.834	13.8155
⋮	⋮	⋮					
1000000000	~21.3004	~3.05872	0.999999999	-9.9999997222×10^{-10}	326934142.38635	19.6052693	20.723265837

We can further simplify the relation,

or, $x + \ln\left(\frac{n-1}{n}\right) - \ln\frac{x}{n} \le e^x$

or, $x + \ln\left(\frac{n-1}{n}\cdot\frac{n}{x}\right) \le e^x$

or, $x + \ln\frac{(n-1)}{x} \le e^x$

or, $\frac{x}{e^x} + \frac{1}{e^x}\ln\frac{(n-1)}{x} \le 1$ (step L)

Here, $x \ll e^x$ and $\ln\frac{(n-1)}{x} < \ln\frac{n}{x} < \ln n < e^x = H_n$

So, $\frac{x}{e^x} \ll 1$ and $\frac{1}{e^x}\ln\frac{(n-1)}{x} < \frac{1}{e^x}\ln\frac{n}{x} < \frac{1}{e^x}\ln n < 1$.

In the (step L) of the relation, with the increase in n , and consequently, *x*, there would be more and more significant changes in $\frac{x}{e^x}$ and $\frac{1}{e^x}$ reducing the value of left hand side by division by larger and increasingly larger denominator (exponential divisor)compared to numerator, while the value of $\ln\frac{(n-1)}{x}$ will not change much compared to e^x(as e^xwill change exponentially), will remain close to $\ln n$, always less than e^x, and $\frac{1}{e^x}\ln\frac{(n-1)}{x}$ always less than 1. So, the (step L) is more and more likely to be true for larger and larger numbers, as it represents Lagarias's equivalent of Riemann hypothesis which has been tested and proved for the first 10000000000000 solutions, the (step L) is true.

So, the condition,

$\boldsymbol{\frac{x}{e^x} + \frac{1}{e^x}\ln\frac{(n-1)}{x} \le 1}$**is true.**

It may look more convincing to some if we rewrite (step L) as,

$\frac{x}{e^x} + \frac{1}{e^x}\ln(n-1) - \frac{1}{e^x}\ln x \le 1$ or, $\frac{x-\ln x}{e^x} + \frac{1}{e^x}\ln(n-1) \le 1$

Here, $\frac{x-\ln x}{e^x}$ will be increasingly and exceedingly small fraction when *n* increases, and as $\frac{1}{e^x}\ln n < 1$ so, $\frac{1}{e^x}\ln(n-1) < 1$ and its change would be negligible even after medium-large *n*, compared to the change in $\frac{x-\ln x}{e^x}$.

(e.g.when*n*=10000 then $\ln 10000 = 9.210340371976182736071965818737$5 and $\ln 9999 = 9.2102403669758493777366323187232$ divide each of them by $e^x = H_n = H_{10000}$ =~9.78761, and difference would be very small one, get smaller and smaller and smaller, here 1.0217509722328365692288517247827×10^{-5} and $\frac{1}{e^x}\ln(n-1)$ in this case would be 0.94101015130106832799188283132687 and $\frac{x-\ln x}{e^x}$ would be 0.14880583202400649822138463507075 then $\frac{x-\ln x}{e^x} + \frac{1}{e^x}\ln(n-1)$ =1.0898159833250748262132674663976, apart from any inaccuracy for rounding up the figure we should bear in mind that we are calculating it with that form of the relation when the left hand side takes the highest value, and the relation is least likely to be true; even then, like prime number theorem, the relation would be true for higher values of *n*, as we see when *n*=1000000000 then with values taken from the table1,

$\frac{x-\ln x}{e^x} + \frac{1}{e^x}\ln(n-1) =$ 0.09111206701289313200293330861134 + 0.97290500816634481773402955305186 =1.0640170751792379497369628616632,

when *n*=1000000000000000then,

$\frac{x-\ln x}{e^x}+\frac{1}{e^x}\ln(n-1)=$ 0.98106535988932156658865977456679+0.07174429771529358473266707576945= 1.0528096576046151513213268503362, the value is decreasing on the proposed lesser side of the relation, and will be true for very large values of *n*, and certainly when $n\to\infty$)
So, Lagarias's equivalent of Riemann hypothesis is true even when the left hand side takes the greatest value.
So, **$\sigma(n)\leq H_n + e^{H_n}\ln H_n$ is truewhen $n\to\infty$.**
Consequently, Riemann hypothesis, which has been proved for the first 10000000000000 solutions, is true for all zeros beyond it, for values towards infinity. ■

IV. Conclusion

Riemann hypothesis is one of the most elusive unsolved problems of today. For long mathematicians have tried to prove this problem. There has been effort to devise an elementary version of the problem proof of which indirectly proves Riemann hypothesis. J. C. Lagarias in 2002 published such an elementary version of the hypothesis which has been widely accepted as an elementary equivalent of Riemann hypothesis. In this article there has been an effort to prove Lagarias's condition which consequently proves Riemann hypothesis.

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